ON THE USE OF LIE GROUPOIDS IN FLUID-STRUCTURE INTERACTIONS

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ABSTRACT. Thanks to the work of V.I. Arnold *Sur la géometrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaites* (1966) we know that an ideal fluid can be understood as a Lagrangian mechanical system on a Lie group of volume preserving diffeomorphisms. Moreover, one can reduce by the particle relabeling symmetry of the system to arrive at equations of motion on the Lie algebra of divergence free vector fields. In this article we will investigate the equations of motion for a flexible body immersed in an incompressible ideal fluid. We will find that the configuration manifold for such a system is the source fiber of a Lie groupoid. Moreover, we will reduce by the particle relabeling symmetry of the system to obtain equations of motion on the Lie algebroid in close analogy with the work of V.I. Arnold. As this groupoid acts transitively on itself, we will be able to execute this reduction as an instance of Lagrange-Poincaré reduction. We will also address the case of Navier-Stokes fluids by including a dissipative viscous force and amending the original Lie groupoid to include a no-slip condition.

CONTENTS

1. INTRODUCTION

Since the work of [Arn66], it is well known that the dynamics of fluids can be described as an ODE on a Lie group. Fluid-structure interactions present an interesting challenge because the structure immersed in the fluid interferes with the group properties of the system. In this paper we will preserve as much of this group structure as possible by describing the system on a Lie groupoid. Compelling reasons for doing this are motivated by history. In particular, the publication of [Arn66] was soon followed by:

- proofs of local-time existence of solutions for inviscid fluid equations [EM70]

- the creation of computationally tractable Lie group integrators for fluids [GMP+11]
- the geometrization of complex fluids [Hol02, GBR09]
- a new paradigm of analysis for the Lagrangian stability of fluid motions [Luk88, Mis93].

See [AK92] and [MR99, §13.7] for an overview. The main theorem of this paper parallels that of [Arn66], and we hope to promote a similar research programme for fluid-structure problems. Some specific advantages of having a geometric derivation of fluid-structure equations are:

- We do not make any assumptions on the nature of the immersed body, which could be a rigid or flexible medium, or even another fluid.
- Along the way, we obtain a reduced variational principle for fluid-structure interactions. Such principles are a first step in constructing variational-geometric numerical integrators (see [HLW02, MW01, MV91, BS99]).
- Having a geometric understanding of fluid-structure interactions might prove to be useful in the context of nonlinear stability and control theory. In [Jac12], for instance, the geometric framework established in this paper is used to demonstrate the existence of (approximate) limit cycles for bio-locomotion.

In summary, a diversity of perspectives is often beneficial. We hope the content of this paper provides a new perspective on fluid structure interaction, and opens the door for new types of collaboration between differential geometers, engineers, and applied mathematicians.

1.1. **Main contributions.** The over-arching contribution of this paper is an understanding of fluid-structure mechanical systems using natural operations on Lie groupoids. Many of these ideas will look different from the traditional approach to fluids. Therefore, this paper also provides a "sanity check", where we make sure that the equations of motion in our geometric framework correspond to the standard equations of motion in terms of vector-calculus. In particular, the main contributions can be listed as:

- We will identify the configuration manifold for fluid-structure systems as the source-fiber of a Lie groupoid.
- We will identify the particle relabeling symmetry of a fluid-structure systems as the isotropy group of the same Lie groupoid.
- We will state the equations of motion as on ODE on a Lie algebroid.
- We will verify that the equations of motion stated correspond to the equations of motion used in the engineering literature.

In the following subsections we will review the equations of motion for a structure immersed in a Newtonian fluid and provide a preview of the Lie groupoid which we will be studying throughout the rest of the paper.

1.2. **Fluid-structure interactions.** The equations that determine the motion of a coupled fluid-structure system consist of the Navier-Stokes equation together with a forced Euler-Lagrange equation for the evolution of the body. Specifically, these equations are given by

$$\frac{D}{Dt}\left(\frac{\partial L_{\mathcal{B}}}{\partial \dot{b}}\right) - \frac{\partial L_{\mathcal{B}}}{\partial b} = f_{\mathcal{B}}, \tag{1}$$

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = \nu \nabla^2 u - \nabla p, \quad \nabla \cdot u = 0 \tag{2}$$

Here, ν is the viscosity coefficient, while u and p are the Eulerian velocity and the pressure field, $L_{\mathcal{B}}$ is the Lagrangian of the body and $f_{\mathcal{B}}$ is the force exerted by the fluid on the body, given by the expression for virtual work

$$f_{\mathcal{B}} \cdot \delta b = -\int_{\partial \mathcal{B}} \delta b \cdot \mathbf{T} \cdot n \, dS \tag{3}$$

where $\mathbf{T} = -p\mathbf{I} + \nu\left(\nabla u + \nabla u^T\right)$ is the stress tensor of the fluid and $n(x)$ is the unit external normal to the boundary of the body. For the case of a rigid body immersed in a viscous fluid, these equations can be found in, for instance, [MZ05]. For the case of general structures immersed in viscous fluids, one can deduce these equations from the standard theory of continuum mechanics using the stress tensor $\mathbf{T}$, [Bat00, §3.3, §5.14].

In this paper, we will consider both the case of a viscous fluid interacting with a body as well as the inviscid case, for which $\nu = 0$ in (2). The boundary conditions will be chosen accordingly and are discussed below.

1.3. **Lie groups and groupoids in fluid mechanics.** In his foundational work, [Arn66] showed that the motion of an inviscid, incompressible fluid moving in a fixed container M describes geodesic curves on the group $\mathrm{SDiff}(M)$ of volume-preserving diffeomorphisms of M. As mentioned before, this geometric insight gave rise to a plethora of applications and new results, but is not immediately applicable to the case of fluids interacting with other media: as the body moves through the fluid, the fluid container changes from one instance to the next, so that the fluid configurations can no longer be described by diffeomorphisms of a single, fixed manifold.

A first contribution of this paper is to show that fluid-structure interactions can be described by means of a *Lie groupoid* instead of a Lie group. To explain this concept, we recall that the configuration of an immersed body can be described by means of embeddings $b : \mathcal{B} \to \mathbb{R}^d$ where $\mathcal{B}$ is a manifold which represents the body. The fluid space is the complement of the space occupied by the body, which we denote by

$$\approx\!\!\approx_b := \text{closure}\left(\mathbb{R}^d \backslash b(\mathcal{B})\right).$$

Now, if the rigid body moves from one configuration, b_0, to another, b_1, the fluid container changes from $\approx\!\!\approx_{b_0}$ to $\approx\!\!\approx_{b_1}$. The configuration of the fluid, meanwhile, is

given by a (volume-preserving) diffeomorphism $\varphi : \approx\approx_{b_0} \to \approx\approx_{b_1}$. This motivates us to introduce the set

$$G = \{(b_1, \varphi, b_0), \quad b_0, b_1 \in \mathrm{Emb}(\mathcal{B}), \varphi \in \mathrm{SDiff}(\approx\approx_{b_0}, \approx\approx_{b_1})\} \tag{4}$$

as the natural configuration space for fluid-structure interactions, where the set $\mathrm{SDiff}(\approx\approx_{b_0}, \approx\approx_{b_1})$ consists of all volume preserving diffeomorphisms from $\approx\approx_{b_0}$ to $\approx\approx_{b_1}$.

The set G is equipped with a multiplication, which is only partially defined: two elements (b_1, φ, b_0) and (b_1', φ', b_0') can be multiplied only if $b_1 = b_0'$, in which case

$$(b_1', \varphi', b_0') \cdot (b_1, \varphi, b_0) = (b_1', \varphi' \circ \varphi, b_0). \tag{5}$$

As the multiplication operation is defined only for certain pairs of elements, G is not a group like SDiff, but rather a groupoid which is a well studied object within differential geometry. For a gentle introduction to Lie groupoids in geometry we refer to the survey article [Bro87], while a comprehensive, geometric overview can be found in [Mac05]. More information about Lie groupoids in mechanics can be found in [MMdDM06, Wei95] while an application to general relativity using a groupoid similar to G can be found in [BFW09].

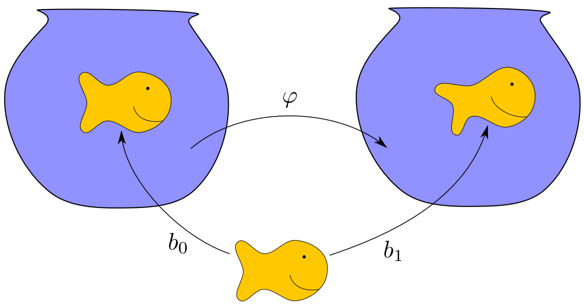

FIGURE 1. The fluid-structure Lie groupoid G consist of triples (b_1, φ, b_0), where b_0, b_1 are embeddings from the body reference configuration into $\mathbb{R}^d$, and φ is a diffeomorphism from $\approx\approx_{b_0}$ to $\approx\approx_{b_1}$. Here $\approx\approx_{b_0}$ is the space available to the fluid when the body is in the configuration b_0, and similarly for $\approx\approx_{b_1}$.

The infinitesimal version of a Lie groupoid is referred to as a Lie algebroid (see Figure 2), and in this paper, we will show that the Lie algebroid $\mathcal{A}$ of (4) consists of all triples $(b, \dot{b}, u)$, where $b \in \mathrm{Emb}(\mathcal{B})$ and $\dot{b} \in T_b \mathrm{Emb}(\mathcal{B})$ are a body configuration and velocity, and u is an (Eulerian) fluid velocity field. Both the groupoid G as well as its algebroid $\mathcal{A}$ therefore have clear and straightforward interpretations in terms of fluid dynamics.

1.4. **Derivation of the fluid-structure equations.** The second part of our paper is devoted to the derivation of the fluid-structure equations (1, 2) using the framework of *Lagrange-Poincaré reduction*. In this paper, we start from geodesic

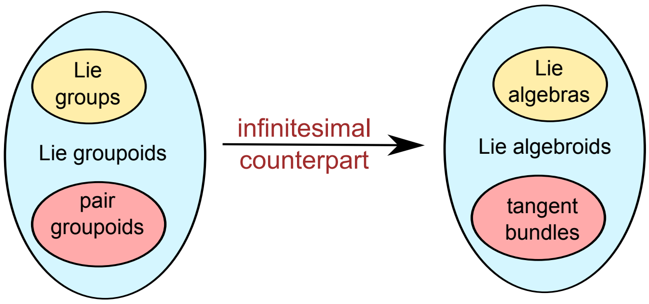

FIGURE 2. Lie groups and pair groupoids are special cases of Lie groupoids. Their infinitesimal counterparts are Lie algebras and tangent bundles, respectively, and form examples of Lie algebroids.

motion on the groupoid G and we observe that this system has the group of particle relabelings as its symmetry group. This is an extension of Arnold's original observation in [Arn66] to the groupoid context. Factoring out by this symmetry, we obtain a reduced system, which can be given a particularly convenient description once we choose a horizontal subspace of the Lie algebroid. In the last part of the paper, we include viscous dissipation into the fluid-structure picture, by modeling it as a (non-conservative) force.

1.5. **Previous work.** This paper provides a bridge between two areas of study: fluid structure interaction and the theory of Lie groupoids. As a result we draw from a range of sources on both sides of this divide. In particular we will exploit the fact that the Lagrange-Poincaré equations can be seen as equations on a Lie algebroid. There is much work in the world of fluid structure interaction which hints at viewing fluid-structure systems in the light of Lagrange-Poincare reduction. Specifically, it is well-known that fluid-structure systems exhibit fluid particle relabeling symmetry, and one could surmise that reducing by this symmetry yields Lagrange-Poincaré equations. However, to the best of our knowledge, this derivation does not yet exist in the literature. Nonetheless, similar tasks have been executed, and this "gauge theoretic" flavor to fluid mechanics has been adopted in a number of instances. For example, [SW89] developed a gauge theory for swimming at low Reynolds numbers. This gauge theoretic picture of swimming was then applied to potential flow in [Kel98]. Building upon [Kel98], the equations of motion for an articulated body in potential flow were derived as Lagrange-Poincaré equations (with zero momentum) in [KMRMH05] and then expanded to handle point vortices in [VKM09]. Similarly, [Rad03] generalized the work of [Kel98] to arbitrary flows and and adopted the Lagrange-Poincaré equations as valid equations of motion.

On the other side of the main theorem for this paper lies the world of Lie groupoids and geometric mechanics. The equations of motion we derive will

be achieved via a reduction by symmetry of a Lagrangian mechanical system. Early work on reduction of Lagrangian systems by symmetry can be found in [MS93b, MS93a] with an intrinsic formulation articulated in [CMR01] for the case of arbitrary finite dimensional configuration manifolds and left-invariant group symmetries. Around the same time, a notion of Lagrangian mechanics on Lie algebroids was born in [Wei95] and expanded to handle degenerate Lagrangians in [Mar01]. It was made clear in [Wei95] that the Lagrange-Poincaré equations correspond to the Euler-Lagrange equations on a Lie algebroid when the Lie algebroid is transitive. The theory of reduction by symmetries on Lie algebroids was developed further in [Mes05].

1.6. **Organization of the paper.** In §2 we will cover preliminary material to be used in proofs and definitions to follow. The reader is encouraged to read this section casually and refer back to it as needed. In section §3 we will define the fluid-structure groupoid, G, and its Lie algebroid $\mathcal{A}$. We will also describe the configuration manifold $Q(b_0)$ as a source fiber of G. Finally we will find that $\mathcal{A} = TQ(b_0)/G(b_0)$ where $G(b_0)$ is a Lie group. Having laid down the kinematics we will be able to construct the Lagrangian on the tangent bundle $TQ(b_0)$ in section §4. We will find that the Lagrangian exhibits a particle relabling symmetry, represented by the Lie group $G(b_0)$ and this will suggest that the equations of motions can be written on $\mathcal{A}$. In order to perform this reduction from dynamics on $TQ(b_0)$ to dynamics on $\mathcal{A}$ we will need to understand variations on $\mathcal{A}$ better, and so we study this in §5. Using this understanding of variations on $\mathcal{A}$ we will be able to transfer Hamilton's variational principle and derive reduced equations of motion on $\mathcal{A}$ in §6 using Lagrange-Poincaré reduction. These equations of motion are written in two ways and provide equivalent descriptions of the same system from alternative perspectives: one which is more geometric (Theorem 6.1) while the other description is the traditional description of fluid structure interactions (Theorem 6.2). The proof of Theorem 6.2 is long and relegated to Appendix B. Finally in §7 we will add viscous dissipation and a no-slip condition to derive the equations of motion for a body in a viscous fluid using the same formalism.

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2. PRELIMINARY MATERIAL

In this section we will recall some notions from differential geometry which will be used in the remainder of the paper. We will assume the reader is familiar with manifolds and vector bundles, and we refer to [KN63, KSM99] for a comprehensive overview. All maps will be assumed smooth throughout this paper. Given any manifold, M, its tangent bundle will be denoted by $\tau_M : TM \to M$ or by TM for short. Additionally, given a map $f : M \to N$, the tangent lift will be given by $Tf : TM \to TN$. We denote the set of k-forms on M by $\Omega^k(M)$. Given a vector bundle $\pi_E : E \to M$ we denote the set of E-valued k-forms by $\Omega^k(M; E) \equiv \Gamma(E) \otimes \Omega^k(M)$. If $\alpha \in \Omega^k(M)$, the exterior derivative of α will be denoted $d\alpha$ and similarly we may define the exterior derivative of a E-valued form by $d(e \otimes \alpha) = e \otimes d\alpha$ where $e \in \Gamma(E)$. Lastly, pullback and pushforward by a map, $\varphi : M \to N$, will be denoted by φ^* and φ_* respectively.

2.1. **Vertical bundles.** Let $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ be vector bundles over M. We define the Whitney sum

$$E_1 \oplus E_2 := \{(e_1, e_2) \in E_1 \times E_2 | \pi_1(e_1) = \pi_2(e_2)\}$$

which is itself a vector-bundle over M equipped with the addition operator

$$(e_1, e_2) + (f_1, f_2) = (e_1 + f_2, e_2 + f_2)$$

for any two $(e_1, e_2), (f_1, f_2) \in E_1 \oplus E_2$ contained in the same fiber. For any vector bundle, $\pi : E \to M$, we may invoke the tangent bundle, $\tau_E : TE \to E$, and define the vertical bundle, $\pi_V : V(E) \to E$, where

$$V(E) = \text{kernel}(T\pi)$$

and $\pi_V := \tau_E|_{V(E)}$. Equivalently, we may define the vertical bundle as the set of tangent vectors of E of the form $\frac{d}{d\epsilon}\big|_{\epsilon=0} (e + \epsilon f)$ where f, e are in the same fiber. This gives us the following proposition.

Proposition 2.1. *Let $v^\uparrow : E \oplus E \to V(E)$ be the map*

$$v^\uparrow(e_1, e_2) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} (e_1 + \epsilon e_2) .$$

Then $v^\uparrow$ is an isomorphism between $V(E)$ and $E \oplus E$. We call $v^\uparrow$ the vertical lift.

2.2. **Connections.** There are three types of connections used in this paper: Levi-Civita connections, connections on a vector bundle, and principal connections. In this section, we will review the first two notions while principal connections will be discussed in Appendix A. In particular, we define the covariant derivative induced by a vector bundle connection, as this concept will be used to formulate the equations of motion throughout the paper; see for instance theorems 6.1 and 6.2.

Definition 2.1. *A* horizontal sub-bundle *on a vector bundle, $\pi : E \to M$, is a subbundle, $H(E) \subset TE$, such that $TE = H(E) \oplus V(E)$. Given a horizontal sub-bundle we define the projections* $\mathrm{ver} : TE \to V(E)$ *and* $\mathrm{hor} : TE \to H(E)$*. The projection,* ver*, is called a* connection*.*

This definition of connection appears in [KSM99] where it is called a "generalized connection". The choice of a connection has a number of consequences, the most important being the existence of a horizontal lift.

Proposition 2.2. *Given a connection on a vector bundle $\pi : E \to M$, there exists an inverse to the map $\tau_E \oplus T\pi|_{H(E)} : H(E) \to E \oplus TM$. We denote this inverse by $h^\uparrow : E \oplus TM \to H(E)$, and call it the* horizontal lift*.*

Proof. Choose an $e \in E$ and let $H_e(E), V_e(E), T_eE$ denote the fibers of the horizontal bundle, the vertical bundle, and the tangent bundle above e. As $T_eE = H_e(E) \oplus V_e(E)$ and $V_e(E) = \mathrm{kernel}(T_e\pi)$, we see that $T_e\pi|_{H_e(E)}$ is injective. Additionally, $T_e\pi|_{H_e(E)}$ must also be surjective since $T_e\pi(T_eE) = T_mM$ where $m = \pi(e)$. Thus $T_e\pi|_{H_e(E)}$ is invertible. We define $h^\uparrow : E \oplus TM \to H(E)$ by $h^\uparrow(e, \dot{m}) = T_e\pi|_{H_e(E)}^{-1}(\dot{m})$. By construction $h^\uparrow \circ \tau_E \oplus T\pi|_{H(E)}$ is the identity on $H(E)$. □

We will occasionally use the notation $h_e^\uparrow(\dot{m}) \equiv h^\uparrow(e, \dot{m})$. Additionally, the choice of a connection induces a covariant derivative.

Definition 2.2. *We define the* vertical drop *to be the map, $v_\downarrow : V(E) \to E$ given by*

$$v_\downarrow = \pi_2 \circ (v^\uparrow)^{-1}$$

where $\pi_2 : E \oplus E \to E$ is the projection onto the second E component. Let $I \subset \mathbb{R}$ be an interval so that we may consider the set $C(I, E)$ consisting of curves from I into E. We define the covariant derivative *with respect to the connection $H(E)$ to be the map, $\frac{D}{Dt} : C(I, E) \to E$, given by*

$$\frac{De}{Dt} = v_\downarrow \left[\mathrm{ver}\left(\frac{de}{dt} \right) \right]$$

for a curve $e(\cdot) \in C(I, E)$.

Lastly, the choice of a connection also provides two partial derivatives on a vector bundle.

Definition 2.3. *Let $\pi : E \to M$ be a vector bundle and let $f : E \to \mathbb{R}$ be a function. Given a connection we define the partial derivative of f with respect to m as the vector bundle morphism, $\frac{\partial f}{\partial m} : E \to T^*M$, given by*

$$\left\langle \frac{\partial f}{\partial m}(e), \delta m \right\rangle := \langle df(e), h_e^\uparrow(\delta m) \rangle;$$

see [AMR09]. *Additionally we define the* fiber derivative, $\frac{\partial f}{\partial e} : E \to E^*$, *given by*

$$\left\langle \frac{\partial f}{\partial e}(e), e' \right\rangle = \langle df(e), v^\uparrow(e, e') \rangle := \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} (f(e + \epsilon e')) .$$

These partial derivatives allow us to write down the differential of a real-valued function in a new way.

Proposition 2.3. *Let $f : E \to \mathbb{R}$. Let $\delta e \in TE$ and $\delta m = T\pi(\delta e)$. Then*

$$\langle df(e), \delta e \rangle = \left\langle \frac{\partial f}{\partial m}(e), \delta m \right\rangle + \left\langle \frac{\partial f}{\partial e}(e), \frac{De(t)}{Dt} \right\rangle .$$

where $e(t)$ is a path in E tangent to δe.

Proof. We observe that

$$\langle df, \delta e \rangle = \langle df, \mathrm{hor}(\delta e) \rangle + \langle df, \mathrm{ver}(\delta e) \rangle.$$

By definition, the first term is identically $\langle df, h^\uparrow(T\pi(\delta e)) \rangle = \left\langle \frac{\partial f}{\partial m}, \delta m \right\rangle$. The second term is

$$\langle df, \mathrm{ver}(\delta e) \rangle = \left\langle df, v^\uparrow(e, \frac{De(t)}{Dt}) \right\rangle = \left\langle \frac{\partial f}{\partial e}, \frac{De(t)}{Dt} \right\rangle$$

□

The covariant derivative on curves in E naturally induces a covariant derivative on curves in E^* given by the condition

$$\frac{d}{dt} \langle \alpha, e \rangle = \left\langle \frac{D\alpha}{Dt}, e \right\rangle + \left\langle \alpha, \frac{De}{Dt} \right\rangle . \tag{6}$$

Remark 2.1. Given vector bundles E_1 and E_2 with horizontal sub-bundles $H(E_1)$ and $H(E_2)$ we can define a horizontal sub-bundle by taking the direct sum: $H(E_1 \oplus E_2) = H(E_1) \oplus H(E_2)$. This will be valuable later when dealing with equations of motion on the direct sum of a Riemannian manifold and an adjoint bundle.

2.3. **Connections on Riemannian manifolds.** The two most important examples of connections (for us) are the Levi-Civita connection of a Riemannian manifold, and a principal connection of a principal bundle. We will review these next. To do this we introduce the notion of an affine connection.

Definition 2.4. *An affine connection on a vector bundle $\pi : E \to M$ is a mapping, $\nabla : \mathfrak{X}(M) \oplus \Gamma(E) \to \Gamma(E)$ such that*

(1) $\nabla_{fX+gY}(\sigma) = f\nabla_X\sigma + g\nabla_Y\sigma$ *for* $f, g \in C^\infty(M), X, Y \in \mathfrak{X}(M), \sigma \in \Gamma(E)$.
(2) $\nabla_X(\sigma_1 + \sigma_2) = \nabla_X\sigma_1 + \nabla_X\sigma_2$ *for* $\sigma_1, \sigma_2 \in \Gamma(E)$.
(3) $\nabla_X(f\sigma) = X[f]\sigma + f\nabla_X\sigma$ *for* $f \in C^\infty(M), X \in \mathfrak{X}(M), \sigma \in \Gamma(E)$.

Let M be a Riemannian manifold with a metric $\langle\cdot,\cdot\rangle_M$. A connection on TM is a subspace, $H(TM) \subset TTM$ and the covariant derivative is a map $\frac{D}{Dt} : C(I, TM) \to TM$. The covariant derivative uniquely defines an affine connection by the equivalence $(\nabla_X(Y))(m_0) = \frac{Dv}{Dt}$ where $v(t) = Y(m(t))$ and $m(t)$ is the integral curve of X with the initial condition m_0 [AM00, §2.7]. There is a one-to-one correspondence between affine connections and covariant derivatives.

On a Riemannian manifold there is a canonical affine connection which satisfies the following two extra properties:

(1) $\nabla_X Y - \nabla_Y X = [X, Y]$ (torsion free).
(2) $\langle\nabla_Z X, Y\rangle_M + \langle X, \nabla_Z Y\rangle_M = Z[\langle X, Y\rangle_M]$ (metric)

for any $X, Y, Z \in \mathfrak{X}(M)$. We call this unique affine connection the Levi-Civita connection.

Proposition 2.4. *Let $\alpha \in \Omega^1(M)$. A torsion-free connection (such as the Levi-Civita connection) on TM induces the equivalence*

$$d\alpha(v, w) = \left\langle \frac{\partial\alpha}{\partial x}(w), v \right\rangle - \left\langle \frac{\partial\alpha}{\partial x}(v), w \right\rangle .$$

where we are viewing α as a linear function from TM to $\mathbb{R}$ on the right hand side, and an element of $\Omega^1(M)$ on the left hand side.

Proof. Let $X, Y \in \mathfrak{X}(M)$. By the definition of $\frac{\partial\alpha}{\partial x}$ viewed as a function on TM we find

$$\left\langle \frac{\partial\alpha}{\partial x}(X), Y \right\rangle (x) = \langle d_{TM}\alpha, h^{\uparrow}_{X(x)}(Y(x))\rangle$$

where d_{TM} is the exterior derivative on TM. Note that $h^{\uparrow}_{X(x)}(Y(x))$ is a vector in TTM over $X(x) \in TM$ obtained by parallel transporting the vector $Y(x)$ over a curve with velocity $X(x)$. In other words

$$\left\langle \frac{\partial\alpha}{\partial x}(X), Y \right\rangle (x) = \left.\frac{d}{dt}\right|_{t=0} \langle\alpha(\tilde{x}(t)), Y_t\rangle$$

where $\tilde{x}(t)$ is an arbitrary curve such that $\tilde{x}(0) = x$ and $\left.\frac{d\tilde{x}}{dt}\right|_{t=0} = X(x)$. By the induced covariant derivative on curves of covectors we find

$$\left\langle \frac{\partial\alpha}{\partial x}(X), Y \right\rangle (x) = \left\langle \left.\frac{D\alpha(\tilde{x}(t))}{Dt}\right|_{t=0}, Y_0 \right\rangle + \left\langle \alpha(x), \left.\frac{DY_t}{Dt}\right|_{t=0} \right\rangle .$$

However, as the Y_t is parallel along $\tilde{x}(t)$ the second term is 0. Therefore

$$\left\langle \frac{\partial\alpha}{\partial x}(X), Y \right\rangle (x) = \left\langle \left.\frac{D\alpha(\tilde{x}(t))}{Dt}\right|_{t=0}, Y_0 \right\rangle .$$

As $Y_0 = Y(x)$ we see that we can use the identity $\frac{d}{dt}\langle \alpha, Y\rangle = \left\langle \frac{D\alpha}{Dt}, Y\right\rangle + \left\langle \alpha, \frac{DY}{Dt}\right\rangle$ to rewrite the above equations as

$$\left\langle \frac{\partial \alpha}{\partial x}(X), Y \right\rangle(x) = \left.\frac{d}{dt}\right|_{t=0} \langle \alpha(x(t)), Y(x(t))\rangle - \left\langle \alpha, \left.\frac{D}{Dt}\right|_{t=0} (Y(x(t))) \right\rangle.$$

As $x \in M$ is arbitrary, we can write this in terms of the affine connection ∇ as

$$\left\langle \frac{\partial \alpha}{\partial x}(X), Y \right\rangle = \mathcal{L}_X \langle \alpha, Y\rangle - \langle \alpha, \nabla_X Y\rangle.$$

If we swap X and Y and subtract these equations we find

$$\left\langle \frac{\partial \alpha}{\partial x}(X), Y \right\rangle - \left\langle \frac{\partial \alpha}{\partial x}(Y), X \right\rangle = \mathcal{L}_X \langle \alpha, Y\rangle - \mathcal{L}_Y \langle \alpha, X\rangle - \langle \alpha, \nabla_Y X - \nabla_X Y\rangle.$$

By the torsion free property of the Levi-Civita connection, this implies

$$\left\langle \frac{\partial \alpha}{\partial x}(X), Y \right\rangle - \left\langle \frac{\partial \alpha}{\partial x}(Y), X \right\rangle = \mathcal{L}_X \langle \alpha, Y\rangle - \mathcal{L}_Y \langle \alpha, X\rangle - \langle \alpha, [X, Y]\rangle.$$

However the right hand side is simply $d\alpha(X, Y)$. □

Note that Proposition 2.4 presupposes that the exterior derivative is well defined. This is never a problem on finite-dimensional manifolds. However, infinite-dimensional manifolds require that we deal with some functional analytic concerns which we would like to skip [AMR09, see §7.4]. However, one can verify that the above theorem is true in the infinite-dimensional case if the exterior derivative exists. So we must add this existence assumption in the infinite dimensional case.

Given a vector bundle $\pi : E \to M$ we may define the set of E-valued forms $\Omega^1(M; E) := \Gamma(E) \otimes \Omega^1(M)$. This allows us to define the exterior differential of an E valued form by $d(e \otimes \alpha) = e \otimes d\alpha$ for an arbitrary $e \in \Gamma(E)$ and $\alpha \in \Omega^1(M)$. It is simple to see that using this definition for the differential of an E-valued form proposition 2.4 holds for E-valued forms as well. Additionally, the choice of a connection (perhaps not a Levi-Civita connection) on TM induces an intrinsic means of writing the Euler-Lagrange equations.

Proposition 2.5. *Let $q(t)$ be a curve in a manifold Q and let $L : TQ \to \mathbb{R}$ be a Lagrangian. Then the following are equivalent*

(1) *The curve, $q(t)$, satisfies the Euler-Lagrange equations*

$$\frac{D}{Dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

with respect to an arbitrarily chosen connection.

(2) *The curve, $q(t)$, extremizes the action*

$$S = \int_0^T L(q, \dot{q}) dt$$

with respect to variations of $q(t)$ with fixed end-points.

Proof. Before we begin we note that we have an equivalence of mixed partials: let $q(t,\epsilon)$ be an embedding of a surface into Q (i.e. a deformation of a curve). Then we observe that

$$\frac{D\dot{q}}{D\epsilon} = v_{\downarrow}\left(\mathrm{ver}\left(\frac{\partial^2 q}{\partial t \partial \epsilon}\right)\right) = \frac{D\delta q}{Dt} \tag{7}$$

where $\dot{q} = \frac{\partial q}{\partial t}$ and $\delta q = \frac{\partial q}{\partial \epsilon}$.

Let $q : [0,T] \hookrightarrow Q$ be a curve in Q and let $q(t,\epsilon)$ be a variation of $q(t)$ with fixed endpoints. That is to say, $q(t,0) = q(t)$ and $q(0,\epsilon) = q(0), q(T,\epsilon) = q(T)$ for any ϵ. Then we find

$$\begin{aligned}
\delta S &= \frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} \int_0^T L((q,\dot{q})(t))dt \\
&= \int_0^T \langle dL, \frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} (q(t),\dot{q}(t))\rangle dt \\
&= \int_0^T \langle \frac{\partial L}{\partial q}, \delta q\rangle + \langle \frac{\partial L}{\partial v}, \frac{D}{D\epsilon}(\dot{q}_\epsilon)\rangle dt \\
&= \int_0^T \langle \frac{\partial L}{\partial q}, \delta q\rangle + \langle \frac{\partial L}{\partial v}, \frac{D}{Dt}(\delta q)\rangle dt \\
&= \left(\frac{\partial L}{\partial v}, \delta q\right)\bigg|_{t=0}^T + \int_0^T \langle \frac{\partial L}{\partial q}, \delta q\rangle - \langle \frac{D}{Dt}\frac{\partial L}{\partial v}, \delta q\rangle dt \\
&= \int_0^T \langle \frac{\partial L}{\partial q}, \delta q\rangle - \langle \frac{D}{Dt}\frac{\partial L}{\partial v}, \delta q\rangle dt.
\end{aligned}$$

As the variation is arbitrary, it must be the case that $\frac{\partial L}{\partial q} - \frac{D}{Dt}\left(\frac{\partial L}{\partial v}\right) = 0$ where $v = \dot{q}$. □

3. KINEMATICS

In this section, we return to the fluid-structure Lie groupoid G, which we discussed briefly in (4). We introduce a vector bundle $\mathcal{A}$, whose elements are pairs of body velocities and Eulerian fluid velocities, and we show that $\mathcal{A}$ is the Lie algebroid associated to the Lie groupoid G. Those who have not seen Lie groupoids before ought not be intimidated, as we will not require the reader to have heavy background knowledge on the topic. However, a knowledge of Lie groups will be helpful because many constructions on Lie groupoids parallel constructions on Lie groups.

3.1. **The fluid-structure groupoid G.**

Recall that the fluid-structure groupoid G is given by

$$G := \{(b_1, \varphi, b_0) \quad | \quad b_0, b_1 \in \mathrm{Emb}(\mathcal{B}), \varphi \in \mathrm{SDiff}(\approx_{b_0}, \approx_{b_1})\}, \tag{8}$$

where $\mathrm{SDiff}(\approx\!\!\approx_{b_0}, \approx\!\!\approx_{b_1})$ is the set of all volume-preserving diffeomorphisms from $\approx\!\!\approx_{b_0}$ into $\approx\!\!\approx_{b_1}$. The groupoid G is equipped with a source projection $\pi_0(b_1, \varphi, b_0) = b_0$ and a target projection $\pi_1(b_1, \varphi, b_0) = b_1$. Two element $(b_2, \psi, b_1), (b_1, \varphi, b_0) \in G$ such that the target of the second matches the source of the first element may be "composed" via the partial multiplication

$$(b_2, \psi, b_1) \circ (b_1, \varphi, b_0) := (b_2, \psi \circ \varphi, b_0).$$

We say that this is a "partial multiplication" because we can not neccesarily multiply any element of G with another, but instead we require that targets and sources match in the appropriate way. Clearly this multiplication is associative and elements of the form (b, Id, b) serves as identity elements. Given these identity elements, the inverse of $(b_1, \varphi, b_0) \in G$ is given by (b_0, φ^{-1}, b_1). The associative property, along with the existence of identities and inverses make G into a groupoid. Moreover, G is a smooth manifold, and the multiplication and inverse operations are smooth so that G is a Lie groupoid.

An element $(b_1, \varphi, b_0) \in G$ can express the state of a body immersed in a fluid after a finite time, but *how do we express infinitesimal motions?* It turns out that the infinitesimal analogue of Lie groupoid G is the *Lie algebroid* $\mathcal{A}$ (see Figure 2), which we now describe.

3.2. **The Lie algebroid $\mathcal{A}$.** To describe the configuration of the fluid-structure system, we begin by choosing an embedding $b_0 \in \mathrm{Emb}(\mathcal{B})$ which represents the location of the body at a chosen, initial time. The state of the fluid together with the structure can then be described by the element $(b_0, \mathrm{Id}, b_0) \in G$, where Id is the identity. As the fluid and structure are set in motion, they describe a trajectory $g(t) = (b(t), \varphi_t, b_0)$ in G, where φ_t is a diffeomorphism from $\approx\!\!\approx_{b_0}$ to $\approx\!\!\approx_{b(t)}$. Note that the source of $g(t)$ remains b_0 for all time, so that we obtain a curve in the source fiber over b_0. The infinitesimal motion of the fluid-structure system is obtained by differentiating this curve at $t = 0$, and so we propose the following space as the space of infinitesimal motions:

$$\mathcal{A}_b = \left\{ \left. \frac{dg}{dt} \right|_{t=0} \in TG \quad | \quad g(0) = Id_b, \pi_0(g(t)) = b \right\}.$$

Interpreted physically we may denote an arbitrary element of $\mathcal{A}_b$ by $(b, \dot{b}, u)$ where $(b, \dot{b}) \in T_b \,\mathrm{Emb}(\mathcal{B})$ and $u \in \mathfrak{X}_{\mathrm{vol}}(\approx\!\!\approx_b)$ is an Eulerian fluid velocity which does not penetrate the moving body. In particular, because the "fluid part" of G consists of volume preserving diffeomorphisms which move the fluid, the vector field u must be divergence free and satisfies the no-penetration boundary condition, i.e

$$\nabla \cdot u = 0, \quad \text{and} \quad \langle \dot{b}(b^{-1}(x)) - u(x), n(x) \rangle = 0 \quad \text{for all } x \in b(\partial \approx\!\!\approx_{b_0}). \tag{9}$$

We now let $\mathcal{A}$ be the disjoint union

$$\mathcal{A} = \bigcup_{b \in \mathrm{Emb}(\mathcal{B})} \mathcal{A}_b$$

which is a vector bundle over $\tau : \mathcal{A} \to \mathrm{Emb}(\mathcal{B})$ called the *Lie algebroid* of G. We recall that we arrived at $\mathcal{A}$ by taking infinitesimal variations of the identity element $(b, Id, b) \in G$. From this point of view, the relation between the groupoid G and the algebroid $\mathcal{A}$ is similar to the relation between a Lie group and its Lie algebra; see Figure 2. In fact this analogy extends furture because $\mathcal{A}$ is equipped with a Lie bracket on sections. A section of $\tau : \mathcal{A} \to \mathrm{Emb}(\mathcal{B})$ is a pair of maps $(X_{\mathcal{B}}, X_{\approx\approx}) : \mathrm{Emb}(\mathcal{B}) \to \mathcal{A}$ where $X_{body} \in \mathfrak{X}(\mathrm{Emb}(\mathcal{B}))$ and $\mathfrak{X}_{\approx\approx}$ is a map which assigns a Eulerian fluid velocity field to each $b \in Emb(\mathcal{B})$. The bracket on sections of $\mathcal{A}$ is given by

$$[(X_{\mathcal{B}}, X_{\approx\approx}), (Y_{\mathcal{B}}, Y_{\approx\approx})](b) = ([X_{\mathcal{B}}, Y_{\mathcal{B}}](b), [X_{\approx\approx}(b), Y_{\approx\approx}(b)]).$$

Lastly, there is one more structure on $\mathcal{A}$ known as the anchor map,

$$\rho : (b, \dot{b}, u) \in \mathcal{A} \mapsto (b, \dot{b}) \in T\,\mathrm{Emb}(\mathcal{B}).$$

This completes our description of $\mathcal{A}$. Note that (as a set) $\mathcal{A}$ appears to be capable of expressing the fluid structure equations in the form of (1) and (2) but may seem far removed from the realm of Lagrangian mechanics on tangent bundles. However, $\mathcal{A}$ will be shown to be the quotient of a tangent bundle of a configuration manifold by a Lie group action. This relationship with a tangent bundle of a configuration manifold is what we will use to connect $\mathcal{A}$ to Lagrangian mechanics.

3.3. **The bundle $TQ(b_0)/G(b_0)$.** Given an initial embedding $b_0 \in \mathrm{Emb}(\mathcal{B})$ we may set the initial condition of the fluid-structure system to be the identity element $(b_0, Id, b_0) \in G$. As stated in the previous section, the physical evolution of the system is given as a trajectory $g(t) = (b(t), \varphi_t, b_0) \in G$ where the source of $g(t)$ is b_0 for all time. In other words, if the initial configuration of the body is b_0 then the system evolves on the fiber

$$Q(b_0) := \pi_0^{-1}(b_0) = \{(b, \varphi, b_0) \in G\} \tag{10}$$

$$\equiv \{(b, \varphi) \quad | \quad \varphi \in \mathrm{SDiff}(\approx\approx_{b_0}, \approx\approx_b)\}. \tag{11}$$

We may alternatively write elements of $Q(b_0)$ as pairs (b, φ) when we want to reduce clutter, or as triples $(b, \varphi, b_0) \in G$ when we want to emphasize that $Q(b_0)$ is a subset of G. As trajectories on $Q(b_0)$ are all physically meaningful we may use $Q(b_0)$ as a configuration manifold for a fluid-structure system and we could implement Lagrangian mechanics on $TQ(b_0)$ in the standard way. However, we will postpone Lagrangian mechanics for now so that we may first describe the particle relabeling group for the fluid. In particular consider the Lie group

$$G(b_0) := \pi_0^{-1}(b_0) \cap \pi_1^{-1}(b_0) \equiv \mathrm{SDiff}(\approx\approx_{b_0}).$$

We see that given an element $(b, \varphi) \approx (b, \varphi, b_0) \in Q(b_0)$ and an element $\psi \approx (b_0, \psi, b_0) \in G(b_0)$ we may use the groupoid multiplication to get the element

$$(b, \varphi) \cdot \psi = (b, \varphi, b_0) \circ (b_0, \psi, b_0) = (b, \varphi \circ \psi, b_0) = (b, \varphi \circ \psi).$$

In this way, we obtain a right $G(b_0)$-action on $Q(b_0)$ which relabels the particles of the fluid. Moreover, this action may be lifted to the tangent bundle $TQ(b_0)$ by taking tangent mappings of the action on $Q(b_0)$. In other words, for any $\psi \in G(b_0)$ we may define its action on $TQ(b_0)$ by the mapping

$$(b, \dot{b}, \varphi, \dot{\varphi}) \in TQ(b_0) \mapsto (b, \dot{b}, \varphi \circ \psi, \dot{\varphi} \circ \psi) \in TQ(b_0).$$

This group action on $TQ(b_0)$ allows us to define an equivalence relation where two velocites $(b, \dot{b}, \varphi, \dot{\varphi}) \in TQ(b_0)$ and $(b, \dot{b}', \varphi', \dot{\varphi}') \in TQ(b_0)$ are said to be equivalent if $b = b'$ and there exists a $\psi \in G(b_0)$ such that $\varphi' = \varphi \circ \psi$ and $\dot{\varphi}' = \dot{\varphi} \circ \psi$. We denote the equivalence class of a velocity $(b, \dot{b}, \varphi, \dot{\varphi}) \in TQ(b_0)$ by $[(b, \dot{b}, \varphi, \dot{\varphi})]$ and we denote the set of these equivalence classes by $TQ(b_0)/G(b_0)$. We won't bother with the technicalities to much, but suffice it to say that the set $TQ(b_0)/G(b_0)$ is a manifold in its own right (see [AM00, Prop 4.1.23]). Moreover the following proposition shows that $TQ(b_0)/G(b_0)$ is equivalent to the Lie algebroid $\mathcal{A}$.

Proposition 3.1. *The map which sends Lagrangian to Eulerian velocities, given by*

$$\Psi : (b, \dot{b}, \varphi, \dot{\varphi}) \mapsto (b, \dot{b}, u), \quad \text{where } u = \dot{\varphi} \circ \varphi^{-1},$$

induces a Lie algebroid isomorphism between the quotient bundle $TQ(b_0)/G(b_0)$ and the Lie algebroid $\mathcal{A}$.

Proof. The space $TQ(b_0)/G(b_0)$ is itself an algebroid, where the anchor map is given by $[(b, \dot{b}, \varphi, \dot{\varphi})] \mapsto (b, \dot{b})$ and the vector bundle projection is $[(b, \dot{b}, \varphi, \dot{\varphi})] \mapsto b$. However, we will only show that Ψ induces a vector bundle isomorphism; the fact that it sends the bracket on the space of section of $TQ(b_0)/G(b_0)$ to that on the space of sections of $\mathcal{A}$ is easily verified.

The map $\Psi : TQ(b_0) \to \mathcal{A}$ given in the statement of the proposition is $G(b_0)$-invariant, since

$$\Psi(b, \dot{b}, \varphi \circ \psi, \dot{\varphi} \circ \psi) = (b, \dot{b}, \dot{\varphi} \circ \varphi^{-1}) = \Psi(b, \dot{b}, \varphi, \dot{\varphi})$$

and hence drops to a quotient map from $TQ(b_0)/G(b_0)$ to $\mathcal{A}$, which we also denote by Ψ. This map is linear on the fibers and is readily shown to be injective. All that remains is therefore to show that Ψ has an inverse which is also fiberwise linear. To construct this inverse, let $(b, \dot{b}, u) \in \mathcal{A}$ and choose an arbitrary φ such that $(b, \varphi) \in Q(b_0)$. Consider now the map Φ which takes $(b, \dot{b}, u) \in \mathcal{A}$ into the equivalence class $[(b, \dot{b}, \varphi, u \circ \varphi)] \in TQ(b_0)/G(b_0)$. A small calculation shows that Φ does not depend on the choice of embedding φ, and that $\Phi \circ \Psi = \mathrm{Id}_{TQ(b_0)/G(b_0)}$ and $\Psi \circ \Phi = \mathrm{Id}_{\mathcal{A}}$, so that Φ is the inverse of Ψ. □

Remark 3.1. The isomorphism in (3.1) between the Lie algebroids $\mathcal{A}$ and $TQ(b_0)/G(b_0)$ can be integrated to a groupoid isomorphism between the groupoid G defined previously and the *action groupoid* $(Q(b_0) \times Q(b_0))/G(b_0)$.

4. MECHANICS

In this section we begin the physical description of fluid-structure interactions. The Lagrangian for fluid-structure interactions is the sum of two parts, one of which is fairly general and describes the immersed body, while the other is the kinetic energy of the fluid. Below, we will first explain the Lagrangian mechanics of a body in a vacuum before describing the mechanics of a body in a fluid.

4.1. **Elastic bodies in a vacuum.** The Lagrangian for an elastic body is a function $L_{\mathcal{B}} : T\,\mathrm{Emb}(\mathcal{B}) \to \mathbb{R}$. We recall that the tangent bundle $T\,\mathrm{Emb}(\mathcal{B})$ consists of all pairs $(b, \dot{b})$ where $b \in \mathrm{Emb}(\mathcal{B})$ and $\dot{b}$ is a vector field along b; i.e. a map from $\mathcal{B}$ to $T\mathbb{R}^d$ such that the diagram

$$\begin{tikzcd} & T\mathbb{R}^d \arrow[d, "\tau_{\mathbb{R}^d}"] \\ \mathcal{B} \arrow[ur, "\dot{b}"] \arrow[r, "b"'] & \mathbb{R}^d \end{tikzcd}$$

commutes.

The kinetic energy is the function on $T\,\mathrm{Emb}(\mathcal{B})$ given by

$$K_{\mathcal{B}}(b, \dot{b}) = \frac{1}{2} \int_{\mathcal{B}} m(x) \|\dot{b}(x)\|^2 d^3x$$

where $m : \mathcal{B} \to \mathbb{R}^+$ is the mass-density of the body in the reference configuration. For an elastic body we could construct the potential energy by defining the *right Cauchy-Green strain tensor*, $\mathbf{C}_b := b_*\langle \cdot, \cdot \rangle_{\mathcal{B}}$. By construction $\mathbf{C}_b$ is symmetric with respect to the metric on $\mathbb{R}^d$ and so we define the potential energy, $U_{le}(\mathbf{C}_b)$, for some real valued function U_{le}. Of course, any function $U(b)$ on the shape of the body may serve as a candidate for the potential energy and choices of potential energy which differ from U_{le} lead to various nonlinear theories of elasticity. In any case the Lagrangian for elasticity is

$$L_{\mathcal{B}}(b, \dot{b}) = K_{\mathcal{B}}(b, \dot{b}) - U(b), \tag{12}$$

and the Euler-Lagrange equations of $L_{\mathcal{B}}$ are the equations of motion for a body moving in a vacuum (see [MH83] for details).

Remark 4.1. More generally, we may allow the Lagrangian of the body to be that of a gas, a rigid mechanical system, or even another fluid. We may even desire to add a surface energy, which would be particularly important if we were modelling bubbles or thin films (however we may opt to use $\mathrm{Emb}(\partial\mathcal{B})$ in the case of modeling a bubble). Again, the specifics of the Lagrangian for the body will not be addressed in this paper.

4.2. **Bodies in fluids.** If the body has configuration $b_0 \in \mathrm{Emb}(\mathcal{B})$ at time $t = 0$ and $b_1 \in \mathrm{Emb}(\mathcal{B})$ at time $t = 1$, then the configuration of the fluid can be described by a volume preserving diffeomorphism from $\approx\approx_{b_0}$ to $\approx\approx_{b_1}$. This tells us that the configuration manifold for the fluid-body system with the initial body configuration b_0 is precisely the manifold $Q(b_0)$ described in equation (11). We can define the kinetic energy of the fluid, $L_{\approx\approx} : TQ(b_0) \to \mathbb{R}$, by

$$L_{\approx\approx}(b, \dot{b}, \varphi, \dot{\varphi}) = \frac{1}{2} \int_{\approx\approx_{b_0}} \|\dot{\varphi}(x)\|^2 d\mathrm{vol}. \tag{13}$$

so that the total Lagrangian is given by

$$L(b, \dot{b}, \varphi, \dot{\varphi}) = L_{\mathcal{B}}(b, \dot{b}) + L_{\approx\approx}(b, \dot{b}, \varphi, \dot{\varphi}). \tag{14}$$

We may now “turn the crank” of Lagrangian mechanics to get equations of motion on $TQ(b_0)$. However, we will first reduce the description of the system by the particle relabeling symmetry to obtain equations of motion on $\mathcal{A} = TQ(b_0)/G(b_0)$.

4.3. **Particle relabeling symmetry.** As in [Arn66], the fluid-structure system is invariant under the action of $G(b_0)$ on $Q(b_0)$ by particle relabelings.

Proposition 4.1. *The Lagrangian, L, of equation* (14) *is symmetric with respect to the right action of* $G(b_0)$ *on* $TQ(b_0)$*.*

Proof. We observe that

$$\begin{aligned} L_{\approx\approx}(b, \dot{b}, \varphi \circ \psi, \dot{\varphi} \circ \psi) &= \int_{\approx\approx_{b_0}} \|\dot{\varphi} \circ \psi\|^2 \, d\mathrm{vol} \\ &= \int_{\psi(\approx\approx_{b_0})} \|\dot{\varphi}\|^2 \det(\psi^{-1}) \, d\mathrm{vol} \\ &= \int_{\approx\approx_{b_0}} \|\dot{\varphi}\|^2 \, d\mathrm{vol} \\ &= L_{\approx\approx}(b, \dot{b}, \varphi, \dot{\varphi}) \end{aligned}$$

where we have used both the change of coordinates formula and the volume preservation property, $\det(\psi) = 1$. Therefore $L_{\approx\approx}$ is invariant with respect to $G(b_0)$. The Lagrangian, $L_{\mathcal{B}}$ is symmetric with respect to $G(b_0)$ in a trivial sense. Therefore the sum $L = L_{\approx\approx} + L_{\mathcal{B}}$ is $G(b_0)$ symmetric. □

For every symmetry of a Lagrangian system there is a conserved quantity via Noether's theorem. As in the case of classical hydrodynamics, we will find that the circulation of the fluid is this conserved quantity.

Proposition 4.2. *Let* $\mathfrak{g}(b_0)$ *denote the set of divergence free vector fields on* $\approx\approx_{b_0}$ *which are tangential to the boundary. For each* $\eta \in \mathfrak{g}(b_0)$ *the circulation*

$$J_\eta(b, \dot{b}, \varphi, \dot{\varphi}) = \int_{\approx\approx_b} \langle \varphi_* \eta, \dot{\varphi} \circ \varphi^{-1} \rangle d\mathrm{vol}$$

is conserved under the Euler-Lagrange equations of the Lagrangian, L, given in equation (14)*.*

Proof. A solution curve, $q(t) = (b, \varphi)(t) \in Q(b_0)$, of the Euler-Lagrange equations extremizes the action

$$S = \int_0^T L(b, \dot{b}, \varphi, \dot{\varphi}) dt$$

with respect to variations with fixed end-points. Consider the variation $\delta q = \delta(b, \varphi) = (0, \varphi \cdot \eta)$ for $\eta \in \mathfrak{g}(b_0)$. Since L is invariant with respect to $G(b_0)$ and we see that this variation comes from the deformation $(b, \varphi \circ \exp(\eta\epsilon))$ we see that $\delta S = 0$. Additionally, we calculate that

$$\delta S = \int_0^T \langle \frac{\partial L}{\partial q}, \delta q \rangle + \langle \frac{\partial L}{\partial \dot{q}} \frac{D\dot{q}}{D\epsilon} \rangle dt.$$

By the equivalence of mixed partials (7) this implies,

$$\begin{aligned}
\delta S &= \int_0^T \langle \frac{\partial L}{\partial q}, \delta q \rangle + \langle \frac{\partial L}{\partial \dot{q}}, \frac{D(\delta q)}{Dt} \rangle dt \\
&= \int_0^T \langle \frac{\partial L}{\partial q} - \frac{D}{Dt} \left(\frac{\partial L}{\partial \dot{q}} \right), \delta q \rangle dt + \langle \frac{\partial L}{\partial \dot{q}}, \delta q \rangle \Big|_{t=0}^{T}.
\end{aligned}$$

The first term vanishes because $q(t)$ is assumed to satisfy the Euler-Lagrange equations. Using that $\delta S = 0$ then implies that $\langle \frac{\partial L}{\partial \dot{q}}, \delta q \rangle$ is a conserved quantity. This is Noether's theorem. We need only unpack the terms. We see that

$$\begin{aligned}
\langle \frac{\partial L}{\partial \dot{q}}, \delta q \rangle &= \int_{\approx_{b_0}} \langle \dot{\varphi}, \varphi \cdot \eta \rangle d\mathrm{vol} \\
&= \int_{\approx_{b}} \langle \dot{\varphi} \circ \varphi^{-1}, \varphi_* \eta \rangle d\mathrm{vol} \\
&= J_\eta(b, \dot{b}, \varphi, \dot{\varphi})
\end{aligned}$$

□

Corollary 4.1 (Kelvin's Circulation Theorem)**.** *Given a closed curve $C(t) \in \approx_{b_0}$ which is advected by the fluid and a vector field v along $C(0)$, the quantity*

$$J_v(b, \dot{b}, \varphi, \dot{\varphi}) = \oint_C \langle \varphi_* v, \dot{\varphi} \circ \varphi^{-1} \rangle d^3 x$$

is conserved by the Euler Lagrange equations of L.

Sketch of the proof. This follows from Proposition 4.2 by choosing a sequence of vector fields which converge to one concentrated on C. See [AK92, Chapter 1] for details. □

Given the $G(b_0)$-symmetry of L we can see that the flow of the system exhibits a symmetry as well (as the flow is derived from L). To be precise, a curve $q(t) \in Q(b_0)$ which satisfies the Euler-Lagrange equations extremizes the action

$$S[q] := \int_0^T L(q, \dot{q})(t)dt.$$

For any $\psi \in G(b_0)$ we know that $L((q,\dot{q}) \cdot \psi) = L(q,\dot{q})$. This implies that the curve $q(t) \cdot \psi \in Q(b_0)$ also extremizes the action so that $q(t) \cdot \psi$ satisfies the Euler-Langrange equations as well. Therefore given one solution we obtain a collection of additional solutions; one for each way of relabeling the fluid particles via some $\psi \in G(b_0)$. However, we are usually not concerned with the labels as they are chosen arbitrarily. In fact, because of the symmetry under particle relabelings there is a well defined flow on $\mathcal{A} = TQ(b_0)/G(b_0)$ which is the space of elements $(b, \dot{b}, u)$, where the Eulerian velocity u is given $u = \dot{\varphi} \circ \varphi^{-1}$, so that u does not depend on the choice of fluid labels.

Because the Euler-Lagrange equations are derived from a Lagrangian, the $G(b_0)$ symmetry of the Lagrangian L should yield Euler-Lagrange equations which are also symmetric with respect to $G(b_0)$. Throughout the remainder of the paper we will move towards a description which expresses this idea and to do this we should recall how we have found the equations of motion in the past. In particular, it is standard to use Hamilton's variational principle to derive the Euler-Lagrange equations on $TQ(b_0)$. This variational principal involved taking variations of curves in $Q(b_0)$ and lifting them to variations in $TQ(b_0)$ by time-differentiation. We desire a variational principle on $\mathcal{A} \equiv TQ(b_0)/G(b_0)$ which is consistent with Hamilton's principle, and so we must understand how variations of curves in $Q(b_0)$ are lifted to yield variations in $\mathcal{A}$ in an analogous manner.

5. VARIATIONS IN $\mathcal{A}$ INDUCED BY VARIATIONS IN $TQ(b_0)$

We may derive the equations of motion on $\mathcal{A}$ in the same way that we derive equations of motion in Lagrangian mechanics, i.e. by extremizing the action integral. However, taking variations of curves in $\mathcal{A}$ is not so simple. Given a curve $(b, \dot{b}, u)(t) \in \mathcal{A}$ a variation intuitively has two components: one component varies b, and the other component varies u. The issue with this perspective is that $\dot{b}$ is coupled to u through the no-penetration boundary condition (9). Moreover, not all variations of u are physically meaningful, as we are only concerned with variations of u which come from variations of fluid particle motions φ.

In this section, we will define the admissible variations on $\mathcal{A}$ as those which satisfy (9) and correspond to variations of fluid particle configurations. The language to to this will be aided by an isomorphism which decomposes these variations into "fluid" and "body" components.

5.1. **Decomposing the variations in $\mathcal{A}$.** Hamilton's principle states that the action, $\int L(q,\dot{q})dt$, is extremized along integral curves of the Euler-Lagrange equations with respect to variations of curves with fixed end points. But what is a variation of a curve in this context? To answer this we first define a deformation of a curve. Let I be an open interval on the real line containing 0. A curve $q : t \in I \mapsto q(t) \in Q$ is merely a mapping of I into Q. A *deformation* of the curve, $q(t)$, is a mapping $(t,\lambda) \in I \times I \mapsto q(t,\lambda) \in Q$ such that $q(t,0) = q(t)$. The *variation* with respect to this deformation is $\delta q := \frac{\partial q}{\partial \lambda}\big|_{\lambda=0}$, which is a vector field along the curve $q(t)$. The action is extremized with respect to arbitrary variations of $q(t)$, but this does not imply extremization with respect to arbitrary variations of $\dot{q} := \frac{dq}{dt}$. Specifically, the variations of $\dot{q}$ which we care about are those which are induced by the variation of q. In other words, we only care about variations of $\dot{q}$ of the form $\frac{d}{dt}\delta q$ (i.e. we assume that $q(t,\lambda)$ is a smooth deformation).

We have found that our system has a symmetry by the Lie group $G(b_0)$, and it is reasonable to assert the existence of a flow on the quotient space $\mathcal{A} \equiv TQ(b_0)/G(b_0)$. However, before we can explore this we must first understand the nature of variations of curves in $Q(b_0)$. Just as the variation of the curve $(q,\dot{q})(t) \in TQ(b_0)$ is not arbitrary, we will find that admissible variations in $\mathcal{A}$ are not arbitrary either.

Moreover, $\mathcal{A}$ can be seen, roughly speaking, as the product of two separate spaces, describing the body and fluid velocities, respectively. In order to express this splitting we may choose to decompose $\mathcal{A}$ into complementary vector bundles $\mathcal{A} = H(\mathcal{A}) \oplus V(\mathcal{A})$ where the vertical space is given canonically by

$$V(\mathcal{A}) := \mathrm{kernel}(\rho) \equiv \{(b,0,\xi) \in \mathcal{A}\},$$

with $\rho : \mathcal{A} \to T\,\mathrm{Emb}(\mathcal{B})$ the anchor map. In words, $V(\mathcal{A})$ is the vector bundle of Eulerian fluid velocity fields which leave the body fixed. This vertical space comes equipped with the structure of a Lie-algebra on each fiber by the Lie bracket $[(b,0,\xi_b),(b,0,\eta_b)] := (b,0,[\xi_b,\eta_b])$. We may define the adjoint map $\mathrm{ad}_{\xi_b} : V_b(\mathcal{A}) \to V_b(\mathcal{A})$ by $\mathrm{ad}_{\xi_b}(b,0,\eta_b) = (b,0,[\xi_b,\eta_b])$. We call ad_{ξ_b} the adjoint map, and we call the dual map $\mathrm{ad}^*_{\xi_b} : V_b(\mathcal{A})^* \to V_b(\mathcal{A})^*$ the *coadjoint map*. The coadjoint map acts on the space of fluid momenta and will be important when we address the mechanical aspects of fluid-structure interaction. In summary, $V(\mathcal{A})$ has a rich and canonically defined structure induced by $\mathcal{A}$. In contrast, there does not exists a natural choice for the horizontal space $H(\mathcal{A})$; we must choose one.

5.2. **The horizontal and vertical spaces.** The only requirement on $H(\mathcal{A})$ is that it is complementary to $V(\mathcal{A})$ which means that we require ρ to be bijective on $H(\mathcal{A})$. Therefore, one way to define an $H(\mathcal{A})$ is by choosing a section $\mathcal{I} : T\,\mathrm{Emb}(\mathcal{B}) \to \mathcal{A}$ which satisfies the property $\rho \circ \mathcal{I} = Identity$. In other words, $H(\mathcal{A})$ is defined by a section of ρ, denoted $\mathcal{I}$, by setting $H = \mathrm{range}(\mathcal{I})$. Conversely, if we are given $H \subset \mathcal{A}$, we may define the section $\mathcal{I} = (\rho|_H)^{-1}$.

For $\mathcal{I} : T\,\mathrm{Emb}(\mathcal{B}) \to \mathcal{A}$ to be a section of ρ means that for each $(b, \dot{b}) \in T\,\mathrm{Emb}(\mathcal{B})$ there is a velocity field u given by $(b, \dot{b}, u) = \mathcal{I}(b, \dot{b})$. In particular, $u \in \mathfrak{X}_{\mathrm{vol}}(\approx_b)$ is a velocity field which satisfies the no-penetration boundary condition (9) since $(b, \dot{b}, u) \in \mathcal{A}$. Thus we may alternatively view $\mathcal{I}$ as a map which assigns $(b, \dot{b}) \mapsto u \in \mathfrak{X}_{\mathrm{vol}}(\approx_b)$ such that u is a velocity field which satisfies the no-penetration boundary condition (see figure 3). This identification of $\mathcal{I}$ as a map to Eulerian velocity fields will reduce clutter in future derivations and so this is how we will interpret $\mathcal{I}$.

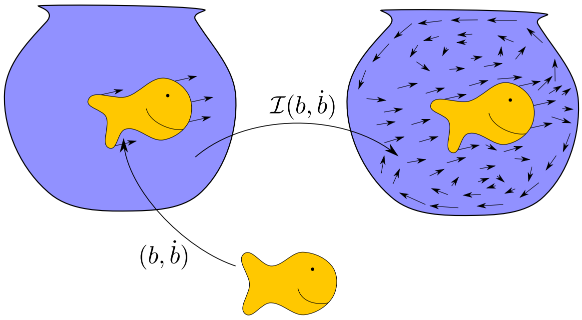

FIGURE 3. The map $\mathcal{I}$ assigns to each body configuration b and body velocity $\dot{b}$ a velocity field $u = \mathcal{I}(b, \dot{b})$ for the fluid, which is such that the no-penetration boundary condition holds.

Remark 5.1. We note that there is considerable freedom in the choice of $\mathcal{I}$, and as we shall see below, all choices of $\mathcal{I}$ yield *different* decomposition into fluid and body velocities but will still give the same equations of motion. The choice of $\mathcal{I}$ is analogous to the choice of a basis for an n-dimensional vector space. Different choices of basis yield different isomorphisms with $\mathbb{R}^n$, and so a particular ODE on a vector space will be written differently on $\mathbb{R}^n$ depending on which basis one chooses.

We now make a small digression to show that one particular choice for $\mathcal{I}$ is singled out on physical grounds. For an inviscid fluid, we begin with the observation that the vorticity is conserved throughout the evolution of the fluid-structure system. As a consequence, the Eulerian velocity field u for the fluid resulting from the body motion given by $\dot{b}$ is be a gradient flow, $u = \nabla\Phi$, where the potential Φ satisfies the following Neumann problem:

$$\Delta\Phi = 0, \quad \text{and} \quad \frac{\partial \Phi}{\partial n}(x) = \left\langle (\dot{b} \circ b^{-1})(x), n(x) \right\rangle \quad \text{for all } x \in b(\partial\mathcal{B}); \tag{15}$$

see [Lam45, Bat00]. The solution of this equation is uniquely defined, up to a constant, and we now define a section of the anchor map denoted $\mathcal{I}_{\emptyset}$ by putting

$$\mathcal{I}_{\emptyset}(\dot{b}) = \nabla\Phi. \tag{16}$$

This section $\mathcal{I}_\varnothing$ can be viewed as the Eulerian version of the "Neumann connection" first introduced in [LMMR86] and further analyzed in [VKM09]. We will return to this observation in Appendix A, where we will show that there is in fact a one-to-one correspondence between the set of sections of $\rho : \mathcal{A} \to T\,\mathrm{Emb}(\mathcal{B})$ and the set of principal connections in $Q(b_0)$.

5.3. **The covariant derivative on $V(\mathcal{A})$.** Let $\mathcal{I}$ be a section of the anchor map viewed as a map from $T\,\mathrm{Emb}(\mathcal{B})$ to Eulerian fluid velocities. Given a curve $\epsilon \mapsto b(\epsilon) \in \mathrm{Emb}(\mathcal{B})$ we may obtain an ϵ-dependent sequence of vector fields

$$u_\epsilon = \mathcal{I}\left(b(\epsilon), \frac{db}{d\epsilon}\right),$$

and φ_ϵ will denote the flow of u_ϵ, i.e. $\frac{d\varphi_\epsilon}{d\epsilon} = u_\epsilon \circ \varphi_\epsilon$. We use the flow φ_ϵ to define a notion of parallel transport on $V(\mathcal{A})$.

Given a curve $b_\epsilon \in \mathrm{Emb}(\mathcal{B})$ we may define the parallel transport of $(b_0, 0, \xi) \in V(\mathcal{A})$ over b_ϵ to be the ϵ-dependent curve $(b_\epsilon, 0, (\varphi_\epsilon)_*\xi) \in V(\mathcal{A})$. This means of parallel transport allows us to define a covariant derivative. Using definition 2.1 we see that one way to define a covariant derivative on the vector bundle $V(\mathcal{A})$ is to define a horizontal sub-bundle to the vector bundle $T(V(\mathcal{A}))$. We can do this by taking the derivative of the parallel transport to get an element

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} (b_\epsilon, 0, (\varphi_\epsilon)_*\xi_\epsilon) = ((b, 0, \xi), (\delta b, 0, \delta\xi - [\mathcal{I}(\delta b), \xi])) \tag{17}$$

where $\delta b = \left.\frac{db}{d\epsilon}\right|_{\epsilon=0}$. The elements of the form (17) define a horizontal space on $T(V(\mathcal{A}))$ and induce the horizontal and vertical projections

$$\begin{aligned} \mathrm{hor}((b,0,\xi),(\dot b,0,\dot\xi)) &= ((b,0,\xi),(\dot b,0,-[\mathcal{I}(\dot b),\xi])) \\ \mathrm{ver}((b,0,\xi),(\dot b,0,\dot\xi)) &= ((b,0,\xi),(0,0,\dot\xi + [\mathcal{I}(\dot b),\xi])). \end{aligned} \tag{18}$$

Proposition 5.1. *The vertical projector given in* (18) *induces the covariant derivative*

$$\frac{D}{Dt}(b, \xi_b) = \left(b, \frac{d\xi_b}{dt} + [\mathcal{I}(\dot b), \xi_b]\right). \tag{19}$$

Proof. This follows from Definition 2.2 for the covariant derivative. □

In Appendix A, we will show that this is in fact the formula for the covariant derivative induced by a principal connection (see [CMR01, JRD12]).

Additionally the covariant derivative on $V(\mathcal{A})$ induces a covariant derivative on the dual vector bundle to $V(\mathcal{A})$ by equation (6). The dual bundle $V(\mathcal{A})^*$ is given by pairs (b, α_b) where α_b is a covector field on $\approx_b$ in the dual space to $\mathfrak{X}_{\mathrm{vol}}(\approx_b)$. and the induced covariant derivative is given by

$$\frac{D(b, \alpha_b)}{Dt} = (b, \dot\alpha_b - \mathrm{ad}^*_{\mathcal{I}(\dot b)}\, \alpha_b). \tag{20}$$

5.4. **The isomorphism between $\mathcal{A}$ and $T\,\mathrm{Emb}(\mathcal{B}) \oplus V(\mathcal{A})$.** Now that we have thouroughly studied $V(\mathcal{A})$, we can construct an isomorphism between the quotient bundle $\mathcal{A}$ and the direct sum $T\,\mathrm{Emb}(\mathcal{B}) \oplus V(\mathcal{A})$. The advantage of this isomorphism is that the summands in the latter space are easier to work with than the original quotient space. From a fluid-dynamical point of view, the isomorphism states the intuitively clear notion that the fluid velocity field can be decomposed into a part consistent with a moving body and a part consistent with a stationary body.

Proposition 5.2. *Given a section of the anchor map, $\mathcal{I}$, the map $\Psi_{\mathcal{I}} : \mathcal{A} \to T\,\mathrm{Emb}(\mathcal{B}) \oplus V(\mathcal{A})$ given by*

$$\Psi_{\mathcal{I}}(b, \dot{b}, u) = (b, \dot{b}, u - \mathcal{I}(\dot{b}))$$

is a vector-bundle isomorphism which preserves the bracket structure on $V(\mathcal{A})$.

Proof. To check that $\Psi_{\mathcal{I}}$ truly maps to the correct space it suffices to check that $u - \mathcal{I}(\dot{b})$ is a vector field which is tangent to the boundary of $\approx_b$. However this is true by construction because both $\mathcal{I}(\dot{b})$ and u must satisfy the no-penetration condition given in equation (9) and so the difference $\mathcal{I}(\dot{b}) - u$ is a vector field which is tangent to the boundary. Additionally, $\Psi_{\mathcal{I}}$ is invertible with the inverse

$$\Psi_{\mathcal{I}}^{-1}(b, \dot{b}, \xi_b) = (b, \dot{b}, \xi_b + \mathcal{I}(\dot{b})).$$

That $\Psi_{\mathcal{I}}$ preserves the brackets on sections is proven by inspection. $\square$

As an illustration, we write down the isomorphism in Proposition 5.2 in the case where the Neumann map defined in (16) is used in place of $\mathcal{I}$. In this case, $\mathcal{I}_{\varnothing}(\dot{b}) = \nabla\Phi$. We then have that

$$\dot{\varphi} \circ \varphi^{-1} = \underbrace{(\dot{\varphi} \circ \varphi^{-1} - \nabla\Phi)}_{\text{consistent with stationary body}} + \underbrace{(\nabla\Phi)}_{\text{consistent with moving body}},$$

where Φ is given by (15). The first factor, $\dot{\varphi} \circ \varphi^{-1} - \nabla\Phi$, is a divergence-free vector field tangent to the boundary, while the second factor is a gradient vector field, so that this decomposition is nothing but the Helmholtz-Hodge decomposition of vector fields. If we let $\mathbb{P}$ denote the projection onto divergence-free vector fields in the Helmholtz-Hodge decomposition, the isomorphism in Proposition 5.2 becomes

$$\Psi_{\mathcal{I}_{\varnothing}}([(b, \dot{b}, \varphi, \dot{\varphi})]) = (b, \dot{b}, \mathbb{P}(\dot{\varphi} \circ \varphi^{-1})).$$

In general, when $\mathcal{I}$ is not a gradient vector field, the isomorphism $\Psi_{\mathcal{I}}$ can be viewed as giving rise to a *generalized Helmholtz-Hodge decomposition*.

5.5. **Covariant variations.** Let $(b, \varphi)_t$ be a curve in $Q(b_0)$. Then a deformation is a λ-parametrized family of curves $(b, \varphi)_{\lambda,t}$ such that $(b, \varphi)_{0,t} = (b, \varphi)_t$. We desire to measure how much the variation $\delta(b, \varphi)_t = \frac{\partial}{\partial\lambda}\big|_{\lambda=0} (b, \varphi)_{\lambda,t}$ changes the quantity $(b, \xi_b)_t := (b_t, \dot{\varphi}_t \circ \varphi_t^{-1} - \mathcal{I}(\dot{b})) \in V(\mathcal{A})$. To do this we invoke the covariant derivative induced by $\mathcal{I}$ of Proposition 5.1.

Additionally, the kinetic energy metric, $\langle\cdot,\cdot\rangle_{\mathrm{Emb}(\mathcal{B})}$, on $\mathrm{Emb}(\mathcal{B})$ induced by the kinetic energy, $K_{\mathcal{B}}$, induces a Levi-Civita covariant derivative on $\mathrm{Emb}(\mathcal{B})$. The Whitney sum of these covariant derivatives is a covariant derivative on the Whitney sum $T\,\mathrm{Emb}(\mathcal{B}) \oplus V(\mathcal{A})$ which we also denote $\frac{D}{Dt}$.

Using this covariant derivative, we may define the *covariant variation* of a curve $(b,\xi_b)_t \in V(\mathcal{A})$ with respect to a deformation $(b,\xi_b)_{\lambda,t}$ by

$$\delta^{\mathcal{I}}(b,\xi_b) := \left.\frac{D(b,\xi_b)}{D\lambda}\right|_{\lambda=0}. \tag{21}$$

We now compute the variation of a reduced curve $(b,\xi_b)_t$ induced by a variation of the curve $(b,\varphi)_t$ in $Q(b_0)$. We split this computation into two parts: in proposition 5.3 we compute the effect of vertical variations, which leave b fixed and act on φ by particle relabelings. In proposition 5.4 we then consider the effect of horizontal variations, which change b (and change φ accordingly).

Proposition 5.3. *Let $\mathcal{I}$ be a section of the anchor map of $\mathcal{A}$ and let $(b,\varphi)_t$ be a curve in $Q(b_0)$. Define the time-dependent vector field*

$$(\xi_b)_t = \dot{\varphi}_t \circ \varphi_t^{-1} - \mathcal{I}(\dot{b}_t),$$

so that $(b,\xi_b)_t \in V(\mathcal{A})$ for all t. Given a vertical deformation of of the curve given by $(b_t, \varphi_t \circ \psi_{t,\lambda})$ for a time-dependent deformation of the identity, $\psi_{t,\lambda} \in G(b_0)$, the covariant variation of $(b,\xi_b)_t$ is given by

$$\delta^{\mathcal{I}}(b,\xi_b) = \frac{D(b,\eta_b)}{Dt} - [(b,\xi_b),(b,\eta_b)]$$

where $\eta_b = (\varphi_t) * \eta$ and $\eta := \left.\frac{\partial\psi_{t,\lambda}}{\partial\lambda}\right|_{\lambda=0} \in \mathfrak{g}(b_0)$.

Proof. To reduce clutter we shall suppress the t and λ dependence of $\varphi_{\lambda,t}$ and abbreviate it as 'φ'. The partial derivative $\delta\xi_b := \frac{\partial}{\partial\lambda}\big|_{\lambda=0}(\xi_b)$ may be split into three parts.

$$\delta\xi_b = \underbrace{\frac{\partial\dot{\varphi}}{\partial\lambda}\circ\varphi^{-1}}_{T_1} + \underbrace{T\dot{\varphi}\circ\frac{\partial\varphi^{-1}}{\partial\lambda}}_{T_2} - \underbrace{\frac{\partial\mathcal{I}(\dot{b})}{\partial\lambda}}_{T_3}.$$

As the variation is vertical, $\mathcal{I}(\dot{b})$ does not depend on λ and therefore $T_3 = 0$. We can rewrite T_1 as

$$\begin{aligned} T_1 = \frac{\partial}{\partial \lambda}(\dot{\varphi}) \circ \varphi^{-1} &= \frac{\partial^2 \varphi}{\partial \lambda \partial t} \circ \varphi^{-1} \\ &= \frac{\partial}{\partial t}(\delta \varphi) \circ \varphi^{-1} \\ &= \frac{\partial}{\partial t}(\varphi \circ \eta) \circ \varphi^{-1} \\ &= T\dot{\varphi} \circ \eta \circ \varphi^{-1} + T\varphi \circ \dot{\eta} \circ \varphi^{-1} \\ &= T\dot{\varphi} \circ \eta \circ \varphi^{-1} + \varphi_* \dot{\eta}, \end{aligned}$$

where we have used the fact that

$$\delta \varphi = \frac{\partial}{\partial \lambda}(\varphi \circ \psi_\lambda)\Big|_{\lambda=0} = T\varphi \circ \eta.$$

On the other hand, T_2 may be written as

$$\begin{aligned} T_2 = T\dot{\varphi} \circ \frac{\partial \varphi^{-1}}{\partial \lambda} &= -T(\dot{\varphi} \circ \varphi^{-1}) \circ \delta\varphi \circ \varphi^{-1} \\ &= -T(\dot{\varphi} \circ \varphi^{-1}) \circ T\varphi \circ \eta \circ \varphi \\ &= -T\dot{\varphi} \circ \eta \circ \varphi^{-1} \end{aligned}$$

Thus we find $\delta \xi_b = \varphi_* \dot{\eta}$. Since $\delta b = 0$, and using the expression (21) for the covariant derivative, we see that

$$\begin{aligned} \delta^{\mathcal{I}}(b, \xi_b) &= (b, \delta \xi_b + [\mathcal{I}(\delta b), \xi_b]) \\ &= (b, \varphi_* \dot{\eta}). \end{aligned}$$

Additionally note that

$$\frac{D(b, \eta_b)}{Dt} = \left(b, \frac{d\eta_b}{dt} + [\mathcal{I}(\dot{b}), \eta_b] \right).$$

We calculate

$$\begin{aligned} \frac{d\eta_b}{dt} = \frac{d}{dt}\varphi_* \eta &= -[\dot{\varphi} \circ \varphi^{-1}, \varphi_* \eta] + \varphi_* \dot{\eta} \\ &= -[\xi_b + \mathcal{I}(\dot{b}), \eta_b] + \varphi_* \dot{\eta}. \end{aligned}$$

Therefore $\varphi_* \dot{\eta} = \frac{d\eta_b}{dt} + [\xi + \mathcal{I}(\dot{b}), \eta_b]$ so that

$$\begin{aligned} \delta^{\mathcal{I}}(b, \xi_b) &= \left(b, \frac{d\eta_b}{dt} + [\xi_b + \mathcal{I}(\dot{b}), \eta_b] \right) \\ &= \frac{D(b, \eta_b)}{Dt} + [(b, \xi_b), (b, \eta_b)]. \end{aligned}$$

□

We understand how variations of curves in $Q(b_0)$ induced by $G(b_0)$ are expressed as variations in $V(\mathcal{A})$, but how about other variations? We must also consider how $(b, \xi_b) = (b, \dot{\varphi} \circ \varphi^{-1} - \mathcal{I}(\dot{b}))$ varies in response to variations of b. Given a curve $(b, \varphi)_t \in Q(b_0)$ we may take a deformation of the curve $b_t \in \mathrm{Emb}(\mathcal{B})$ given by $b_{t,\lambda}$ and define the *horizontal deformation* of $(b, \varphi)_t$ as follows. For fixed t, we consider the vector field $u_{t,\lambda}$ defined by

$$u_{t,\lambda} = \mathcal{I}\left(\frac{\partial b_{t,\lambda}}{\partial \lambda}\right),$$

and we let $\varphi_{t,\lambda}$ be the flow of $u_{t,\lambda}$, so that $\dot{\varphi}_{t,\lambda} = u_{t,\lambda} \circ \varphi_{t,\lambda}$. In this way, we obtain a family of curves $(b_{t,\lambda}, \varphi_{t,\lambda})$, which we will call a horizontal deformation of $(b, \varphi)_t$.

Proposition 5.4. *Given a horizontal deformation of* (b, φ) *given by* $(b_{t,\lambda}, \varphi_{t,\lambda})$ *the covariant variation of* $(b, \xi_b) := (b, \dot{\varphi} \circ \varphi^{-1} - \mathcal{I}(\dot{b}))$ *is*

$$\delta^{\mathcal{I}}(b, \xi_b) = \tilde{B}(\dot{b}, \delta b),$$

where $\tilde{B}$ *is a* $V(\mathcal{A})$*-valued* 2*-form given by*

$$\tilde{B}\left(\dot{b}, \delta b\right) = d\mathcal{I}\left(\dot{b}, \delta b\right) - \left[\mathcal{I}(\dot{b}), \mathcal{I}\left(\delta b\right)\right].$$

Proof. Upon invoking Proposition 2.3 and the fact that the fiber derivative, $\frac{\partial \mathcal{I}}{\partial b}$, is identical to $\mathcal{I}$ (because $\mathcal{I}$ is linear in the fibers) we find that

$$\delta\xi_b = \left.\frac{\partial \dot{\varphi}}{\partial \lambda}\right|_{\lambda=0} \circ \varphi^{-1} - T\dot{\varphi} \circ \varphi^{-1} \circ \delta\varphi \circ \varphi^{-1} - \left\langle \frac{\partial \mathcal{I}}{\partial b}(\dot{b}), \delta b \right\rangle - \mathcal{I}\left(\frac{D}{Dt}(\delta b)\right).$$

Using the equivalence of mixed partials we find that

$$\begin{aligned}
\left.\frac{\partial \dot{\varphi}}{\partial \lambda}\right|_{\lambda=0} \circ \varphi^{-1} &= \frac{d}{dt}(\delta\varphi) \circ \varphi^{-1} \\
&= \frac{d}{dt}\left(\mathcal{I}(\delta b) \circ \varphi\right) \circ \varphi^{-1} \\
&= \left\langle \frac{\partial I}{\partial b}(\delta b), \dot{b} \right\rangle + \mathcal{I}\left(\frac{D}{Dt}(\delta b)\right) + \mathcal{I}(\delta b) \cdot \dot{\varphi} \circ \varphi^{-1}.
\end{aligned}$$

Upon substitution into the last line of the previous calculation we find

$$\delta\xi_b = \left\langle \frac{\partial I}{\partial b}(\delta b), \dot{b} \right\rangle - \left\langle \frac{\partial I}{\partial b}(\dot{b}), \delta b \right\rangle + \mathcal{I}(b, \delta b) \cdot (\dot{\varphi} \circ \varphi^{-1}) - (\dot{\varphi} \circ \varphi^{-1}) \cdot \mathcal{I}(\delta b)$$

and by proposition 2.4

$$\begin{aligned}
&= d\mathcal{I}(\dot{b}, \delta b) + [\mathcal{I}(\delta b), \dot{\varphi}\varphi^{-1}] \\
&= d\mathcal{I}(\dot{b}, \delta b) + [\mathcal{I}(\delta b), \xi_b + \mathcal{I}(\dot{b})].
\end{aligned}$$

Therefore the covariant variation is

$$\delta^{\mathcal{I}}(b, \xi_b) = (b, \delta\xi_b - [\mathcal{I}(\delta b), \xi_b]) = (b, d\mathcal{I}(\dot{b}, \delta b) - \left[\mathcal{I}(\dot{b}), \mathcal{I}(\delta b)\right]) := \tilde{B}(\dot{b}, \delta b). \quad \square$$

In summary, a variation $\delta q(t)$ of a curve $q(t) \in Q(b_0)$ will lead to a variation of the curve $\dot{q}(t) \in TQ(b_0)$ which can be passed to the quotient $\mathcal{A} = TQ(b_0)/G(b_0)$. Moreover if we choose a section of the anchor map, $\mathcal{I}$, we get an isomorphism from $\mathcal{A}$ to the space $T\,\mathrm{Emb}(\mathcal{B}) \oplus V(\mathcal{A})$ and the relevant variations take the form

$$\delta^{\mathcal{I}}(b, \dot{b}, \xi_b) = \left(b, \frac{D\dot{b}}{D\epsilon}, \frac{D\eta_b}{Dt} - [\eta_b, \xi_b] + \tilde{B}(\dot{b}, \delta b)\right).$$

Now that we fully understand the journey from variations of curves in $Q(b_0)$ to variations of curves in $T\,\mathrm{Emb}(\mathcal{B}) \oplus V(\mathcal{A})$ we may use Hamiton's principle on $TQ(b_0)$ to derive the equations of motion on $T\,\mathrm{Emb}(\mathcal{B}) \oplus V(\mathcal{A})$. These equations are known as the *Lagrange-Poincaré equations.*

Remark 5.2. The $V(\mathcal{A})$-valued 2-form $\tilde{B}$ defined in the statement of Proposition 5.4 can be interpreted as the *curvature* of $\mathcal{I}$ when it is related to a principal connection. We will return to this point in Section A.3, but for now we just refer to $\tilde{B}$ as the curvature of $\mathcal{I}$.

6. REDUCED EQUATIONS OF MOTION

In this section we will derive the equations of motion on $T\,\mathrm{Emb}(\mathcal{B}) \oplus V(\mathcal{A})$ using a reduced version of Hamilton's principle known as the Lagrange-Poincaré variational principle. We will first derive the equations of motion in geometric form (Theorem 6.1) and we will then show that these equations are equivalent to the equations (1) and (2) in coordinates (Theorem 6.2).

6.1. **The reduced Lagrangian.** We recall from proposition 4.1 that the total fluid-structure Lagrangian, L, is invariant under the action of $G(b_0)$ on $TQ(b_0)$, so that we may define a reduced Lagrangian $\ell : \mathcal{A} \to \mathbb{R}$ by

$$\ell(b, \dot{b}, u) = L(b, \dot{b}, \varphi, \dot{\varphi}),$$

where φ and $\dot{\varphi}$ are such that $u = \dot{\varphi} \circ \varphi^{-1}$. Explicitely,

$$\ell(b, \dot{b}, u) = L_{\mathcal{B}}(b, \dot{b}) + \frac{1}{2} \int_{\approx\approx_b} \|u(x)\|^2 d\,\mathrm{vol}.$$

In the previous section we found that $\mathcal{A}$ is isomorphic to $T\,\mathrm{Emb}(\mathcal{B}) \oplus V(\mathcal{A})$, given a section of the anchor map $\mathcal{I}$. In terms of $\mathcal{I}$ the reduced Lagrangian on $T\,\mathrm{Emb}(\mathcal{B}) \oplus V(\mathcal{A})$ (which we will also denote by ℓ) is given by

$$\ell(b, \dot{b}, \xi_b) = L_{\mathcal{B}}(b, \dot{b}) + \frac{1}{2} \int_{\approx\approx_b} \|\xi_b + \mathcal{I}(\dot{b})\|^2 d\mathrm{vol},. \tag{22}$$

With this reduced Lagrangian we may transfer Hamilton's principle from $TQ(b_0)$ to the reduced space $T\operatorname{Emb}(\mathcal{B}) \oplus V(\mathcal{A})$ and derive symmetry-reduced equations of motion.

6.2. **The reduced equations of motion.** Now that we understand how variations of curves in $Q(b_0)$ induce variations in $T\operatorname{Emb}(\mathcal{B}) \oplus V(\mathcal{A})$ and that we have an expression for the reduced Lagrangian ℓ, we may transfer Hamilton's principle to the reduced space $T\operatorname{Emb}(\mathcal{B}) \oplus V(\mathcal{A})$. In the following theorem, which is a particular instance of the Lagrange-Poincaré reduction theorem of [CMR01], we derive the equations of motion that result from this reduced variational principle.

Theorem 6.1 (Lagrange-Poincaré Theorem)**.** *Let $\mathcal{I}$ be a section of the anchor map $\rho : \mathcal{A} \to T\operatorname{Emb}(\mathcal{B})$ and let $q(t) = (b, \varphi)_t$ be a curve in $Q(b_0)$. Let $(b, \dot{b}, \xi_b)(t)$ be the induced curve in $T\operatorname{Emb}(\mathcal{B}) \oplus V(\mathcal{A})$, where $\xi_b = \dot{\varphi} \circ \varphi^{-1} - \mathcal{I}(\dot{b})$. Finally, let L be the fluid-structure Lagrangian of equation* (14) *and let ℓ be the reduced Lagrangian of equation* (22)*. Then the following are equivalent.*

(1) *The curve $q(t)$ extremizes the action*
$$S = \int_0^t L(q, \dot{q}) dt$$
with respect to arbitrary variations of $q(t)$ with fixed end points.

(2) *The curve $q(t)$ satisfies the Euler-Lagrange equations*
$$\frac{D}{Dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0.$$

(3) *The curve $(b, \dot{b}, \xi_b)$ extremizes the reduced action*
$$s = \int_0^t \ell(b, \dot{b}, \xi_b) dt$$
with respect to variations, δb, of the curve $b(t) \in \operatorname{Emb}(\mathcal{B})$ with fixed end points and covariant variations of (b, ξ_b) of the form
$$\delta^{\mathcal{I}}(b, \xi_b) = \frac{D(b, \eta_b)}{Dt} + (b, [\xi_b, \eta_b]) + \tilde{B}(\dot{b}, \delta b)$$
for an arbitrary curve $(b, \eta_b)_t \in V(\mathcal{A})$ above b_t.

(4) *The curve $(b, \dot{b}, \xi_b)_t$ satisfies the Lagrange-Poincaré equations:*

$$\frac{D}{Dt}\left(\frac{\partial \ell}{\partial \dot{b}}\right) - \frac{\partial \ell}{\partial b} = i_{\dot{b}} \tilde{B}_\mu \tag{23}$$

$$\frac{D}{Dt}\left(\frac{\partial \ell}{\partial \xi_b}\right) = -\operatorname{ad}^*_{(b, \xi_b)}\left(\frac{\partial \ell}{\partial \xi_b}\right). \tag{24}$$

where $\tilde{B}_\mu$ is a two-form on $\operatorname{Emb}(\mathcal{B})$ given by $\tilde{B}_\mu(\dot{b}, \delta b) = \left\langle \frac{\partial \ell}{\partial \xi_b}, \tilde{B}(\dot{b}, \delta b) \right\rangle$.

Proof. The equivalence of (1) and (2) was established in 2.5. The equivalence of (1) and (3) follows from taking a deformation, $(b,\varphi)_{t,\lambda}$, of $(b,\varphi)_t$ with fixed endpoints. By Proposition 5.3 and 5.4 we see that $\delta^{\mathcal{I}}(b,\xi_b) = \frac{D}{Dt}(b,\eta_b) + (b,[\xi_b,\eta_b] + \tilde{B}(\dot{b},\delta b))$ and δb is a variation of $b(t)$ with fixed endpoints. Therefore, if $q(t)$ extremizes S with respect to arbitrary variations with fixed end points, then clearly $(b,\dot{b},\xi_b)_t$ extremizes s with respect to the variations of the desired form. Conversely, variations δb and $\delta^{\mathcal{I}}(b,\xi_b) = \frac{D}{Dt}(b,\eta_b) + (b,[\eta_b,\xi_b] + B(\dot{b},\delta b))$ produce arbitrary variations of $(b,\varphi)_t$ by the formula $\delta\varphi = (\mathcal{I}(\delta b) + \eta_b)\circ\varphi$. Thus (1) and (3) are equivalent.

Now we will prove the equivalence of (3) and (4). By assuming (3) and taking variations we find

$$
\begin{aligned}
0 &= \delta \int_0^T \ell(b,\dot{b},\xi_b)dt \\
&= \int_0^T \left\langle \frac{\partial \ell}{\partial \xi_b}, \delta^{\mathcal{I}}(b,\xi_b) \right\rangle + \left\langle \frac{\partial \ell}{\partial b}, \delta b \right\rangle + \left\langle \frac{\partial \ell}{\partial \dot{b}}, \frac{D}{D\lambda}(\dot{b}) \right\rangle dt \\
&= \int_0^T \left\langle \frac{\partial \ell}{\partial \xi_b}, \frac{D}{Dt}(b,\eta_b) - [(b,\xi_b),(b,\eta_b)] + \tilde{B}(\dot{b},\delta b) \right\rangle + \left\langle \frac{\partial \ell}{\partial b}, \delta b \right\rangle + \left\langle \frac{\partial \ell}{\partial \dot{b}}, \frac{D}{Dt}(\delta b) \right\rangle dt \\
&= \int_0^T \left\langle \frac{\partial \ell}{\partial \xi_b}, \frac{D}{Dt}(b,\eta_b) - \mathrm{ad}_{(b,\xi_b)}(b,\eta_b) \right\rangle + \left\langle \frac{\partial \ell}{\partial b} - \frac{D}{Dt}\left(\frac{\partial \ell}{\partial \dot{b}}\right) + i_{\dot{b}}\tilde{B}_\mu, \delta b \right\rangle dt \\
&= \int_0^T \left\langle -\frac{D}{Dt}\left(\frac{\partial \ell}{\partial \xi_b}\right) - \mathrm{ad}^*_{(b,\xi_b)}\left(\frac{\partial l}{\partial \xi_b}\right), (b,\eta_b) \right\rangle + \left\langle \frac{\partial \ell}{\partial b} - \frac{D}{Dt}\left(\frac{\partial \ell}{\partial \dot{b}}\right) + i_{\dot{b}}\tilde{B}_\mu, \delta b \right\rangle dt.
\end{aligned}
$$

As η_b and δb are arbitrary we arrive at (4). The above argument is reversible, and thus we have proven equivalence of (3) and (4) as well. □

Of course most people have opted to write the equations of motion for a body immersed in a fluid in the form of equations (1) and (2). While the Lagrange-Poincar'e equations appear different, they are merely same equations in different coordinates. As the proof of this equivalence is rather long, we have relegated it to Appendix B.

Theorem 6.2. *Under the assumptions of Theorem 6.1, the Lagrange-Poincaré equations may be written as*

$$
\frac{D}{Dt}\left(\frac{\partial L_{\mathcal{B}}}{\partial \dot{b}}\right) - \frac{\partial L_{\mathcal{B}}}{\partial b} = F_p \tag{25}
$$

$$
\frac{\partial u}{\partial t} + u\cdot\nabla u = -\nabla p \tag{26}
$$

where $u = \xi_b + \mathcal{I}(\dot{b})$, ∇p *is a Lagrangian parameter which enforces incompressibility, and* $F_p \in T^* \operatorname{Emb}(\mathcal{B})$ *is a force on the boundary of the body given by*

$$\langle F_p, \delta b \rangle = \int_{\partial \approx\!\approx_b} \langle p(x) n(x), (\delta b \circ b^{-1})(x) \rangle d^3 x. \tag{27}$$

7. VISCOUS FLUIDS

To add a viscousity to our formulation we need to alter our constructions in two ways. Firstly, we incorporate the no-slip boundary condition by using the subgroupoid

$$G_{\mathrm{ns}} := \{(b_1, \varphi, b_0) \in G \quad | \quad \varphi \circ b_0|_{\partial \mathcal{B}} = b_1|_{\partial \mathcal{B}}\}$$

where the source and target projections π_0, π_1 are simply the restriction of the original source and target to this subgroupoid. As before we choose a reference configuration $b_0 \in \operatorname{Emb}(\mathcal{B})$ and define the configuration manifold $Q_{\mathrm{ns}}(b_0) = \pi_0^{-1}(b_0)$. The group of fluid particle relabeling symmetries is the group of diffeomorphisms of $\approx\!\approx_{b_0}$ which keep the boundary point-wise fixed,

$$\begin{aligned} G_{\mathrm{ns}}(b_0) &:= \pi_0^{-1}(b_0) \cap \pi_1^{-1}(b_0) \\ &\equiv \{\psi \in \operatorname{SDiff}(\approx\!\approx_{b_0}) \quad | \quad \psi(x) = x, \forall x \in \partial \approx\!\approx_{b_0}\}. \end{aligned}$$

The Lie algebra of $G_{\mathrm{ns}}(b_0)$ is the set

$$\mathfrak{g}_{\mathrm{ns}}(b_0) = \{\eta \in \mathfrak{X}_{\mathrm{vol}}(\approx\!\approx_{b_0}) \quad | \quad \eta(x) = 0, \forall x \in \partial \approx\!\approx_{b_0}\},$$

and the Lie algebroid $\mathcal{A}_{\mathrm{ns}}$ changes accordingly: $\mathcal{A}_{\mathrm{ns}}$ consists of all triples $(b, \dot{b}, u)$ such that $u(x) = \dot{b}(b^{-1}(x))$ for all $x \in \partial \approx\!\approx$. Similarly, the bundle $V(\mathcal{A}_{\mathrm{ns}})$ consists of pairs (b, ξ) such that $\xi = 0$ on the boundary of the domain. It can be proven, as before, that $\mathcal{A}_{\mathrm{ns}}$ is isomorphic to $T \operatorname{Emb}(\mathcal{B}) \oplus V(\mathcal{A}_{\mathrm{ns}})$.

The second change that we need to make is the addition of a viscous friction force. For a viscous fluid with viscosity $\nu > 0$, the viscous force is a map $F_\nu : \mathcal{A} \to \mathcal{A}^*$ defined by

$$\langle F_\nu(b, v_b, u), (b, w_b, v) \rangle = \nu \int_{\approx\!\approx_b} \operatorname{trace}(\nabla u^T \nabla v) d \operatorname{vol} = -\nu \int_{\approx\!\approx_b} \langle \Delta u, v \rangle d \operatorname{vol}.$$

Below, we will also need the lift of F_ν to $TQ(b_0)$, denoted by $\tilde{F}_\nu$ and defined by

$$\left\langle \tilde{F}_\nu(b, v_b, \varphi, v_\varphi), (b, w_b, \varphi, w_\varphi) \right\rangle = \left\langle F_\nu(b, v_b, v_\varphi \circ \varphi^{-1}), (b, w_b, w_\varphi \circ \varphi^{-1}) \right\rangle, \tag{28}$$

for all $(b, v_b, \varphi, v_\varphi), (b, w_b, \varphi, w_\varphi) \in TQ(b_0)$.

Because of the isomorphism between $\mathcal{A}$ and $T \operatorname{Emb}(\mathcal{B}) \oplus V(\mathcal{A})$, we can also view F_ν as a map from $T \operatorname{Emb}(\mathcal{B}) \oplus V(\mathcal{A})$ to $T^* \operatorname{Emb}(\mathcal{B}) \oplus V(\mathcal{A})^*$, denoted by the same letter and given by

$$\langle F_\nu(b, v_b, \xi), (b, w_b, \eta) \rangle = \nu \int_{\approx\!\approx_b} \operatorname{trace} \left(\nabla(\mathcal{I}(v_b) + \xi)^T \nabla(\mathcal{I}(w_b) + \eta) \right) d \operatorname{vol}.$$

Moreover, F_ν can be split as the direct sum of two parts, $F_\nu = F_{\rm ns} \oplus f_\nu$, where $F_{\rm ns} : T\,\mathrm{Emb}(\mathcal{B}) \oplus V(\mathcal{A}) \to T^*\,\mathrm{Emb}(\mathcal{B})$ is the boundary force which will enforce the no-slip condition and $f_\nu : T\,\mathrm{Emb}(\mathcal{B}) \oplus V(\mathcal{A}) \to V(\mathcal{A})^*$ is the viscous force on the fluid. Finally, the force F_ν can be lifted to a force $\tilde{F}_\nu : TQ(b_0) \to T^*Q(b_0)$ in the obvious way. These forces allows us to derive the equations of motion for a body in a Navier-Stokes fluid as shown in the theorem below.

Theorem 7.1. *Let L be the fluid-structure Lagrangian of equation* (14) *restricted to $G_{\rm ns}$ and $\ell : T\,\mathrm{Emb}(\mathcal{B}) \oplus V(\mathcal{A}_{\rm ns}) \to \mathbb{R}$ the reduced Lagrangian written in terms of a section $\mathcal{I} : T\,\mathrm{Emb}(\mathcal{B}) \to \mathcal{A}$ of ρ. Given a curve $q(t) = (b(t), \varphi(t)) \in Q(b_0)$, let $(b, \dot{b}, \xi_b)_t$ be the induced curve in $T\,\mathrm{Emb}(\mathcal{B}) \oplus V(\mathcal{A}_{\rm ns})$, i.e. such that $\xi_b = \dot{\varphi} \circ \varphi^{-1} - \mathcal{I}(\dot{b})$. Then the following are equivalent:*

(1) *The curve, $q(t)$, satisfies the forced Euler-Lagrange equations*
$$\frac{D}{Dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \tilde{F}_\nu,$$
where $\tilde{F}_\nu$ is the lift of the viscous force F_ν, defined in (28).

(2) *The curve $(b, \dot{b}, \xi_b)_t$ satisfies the forced Lagrange-Poincaré equations*
$$\frac{D}{Dt}\left(\frac{\partial \ell}{\partial \dot{b}}\right) - \frac{\partial l}{\partial b} = i_{\dot{b}} \tilde{B}_\mu + F_{\rm ns}$$
$$\frac{D}{Dt}\left(\frac{\partial \ell}{\partial \xi_b}\right) = -\,\mathrm{ad}^*_{(b,\xi_b)}\left(\frac{\partial \ell}{\partial \xi_b}\right) + f_\nu$$
where,
$$\langle F_{\rm ns}(v_b, \xi_b), w_b\rangle = \nu \int_{\partial {\approx\!\approx}_b} \langle n(x) \cdot \nabla u(x), w_b(b^{-1}(x))\rangle d^2x$$
$$\langle f_\nu(v_b, \xi_b), \eta_b\rangle = -\nu \int_{{\approx\!\approx}_b} \langle \Delta(\mathcal{I}(v_b) + \xi_b), \eta_b\rangle d\,\mathrm{vol}.$$

(3) *The curve $(b, \dot{b}, \xi_b)_t$ satisfies*
$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p + \nu \Delta u$$
$$\frac{D}{Dt}\left(\frac{\partial L_{\mathcal{B}}}{\partial \dot{b}}\right) - \frac{\partial L_{\mathcal{B}}}{\partial b} = F_p + F_{\rm ns}$$
where $u = \xi_b + \mathcal{I}(\dot{b})$, and F_p is the pressure force on the boundary of the body given in Theorem 6.1.

Proof. The equivalence of (1) and (2) is merely a statement of the Lagrange-Poincaré reduction theorem with the addition of a force. Assume (2) and we will prove equivalence with (3). We will deal with the vertical equation first by

understanding the force on the fluid. We find that the total viscous force is given by

$$\begin{aligned}
\langle F_\nu(b,\dot b,\xi_b),(b,w_b,\eta_b)\rangle &= \int_{\approx\approx_b} \mathrm{trace}(\nabla u^T \nabla(\eta_b + \mathcal{I}(w_b)))d\,\mathrm{vol} \\
&= \nu \int_{\approx\approx_b} \frac{\partial u^i}{\partial x^j}\left(\frac{\partial \eta^i}{\partial x_j} + \frac{\partial}{\partial x_j}\left[\mathcal{I}(w_b)\right]\right) d\,\mathrm{vol} \\
&= -\nu \int_{\approx\approx_b} \frac{\partial^2 u^i}{\partial (x^j)^2}\left(\eta^i + \mathcal{I}(w_b)^i\right) d\,\mathrm{vol} \\
&= -\nu \int_{\approx\approx_b} \langle \Delta u, \eta + \mathcal{I}(w_b)\rangle d\,\mathrm{vol} \\
&\quad + \nu \int_{\partial\approx\approx_b} (n(x)\cdot\nabla u)\cdot(\mathcal{I}(w_b))d^2S
\end{aligned}$$

Using the boundary condition , $\mathcal{I}(w_b)|_{\partial\approx\approx_b} = w_b \circ b^{-1}|_{\partial\approx\approx_b}$, we may re-write the above force as

$$\begin{aligned}
\langle F_\nu(v_b,\xi_b),(w_b,\eta_b)\rangle &= \nu \int_{\partial\approx\approx_b} \langle n(x)\cdot\nabla u(x), w_b(b^{-1}(x))d^2S \\
&\quad + \int_{\approx\approx_b} \langle \Delta u(x), w_b(b^{-1}(x))\rangle d\,\mathrm{vol}
\end{aligned}$$

Following the same procedure as for the inviscid case (see "step 1" of Appendix B) we arrive at the equation

$$\dot u^\flat + \mathrm{ad}^*_u(u^\flat) = \nu\Delta u.$$

on $\approx\approx_b$. On $\mathbb{R}^d$ this becomes

$$\frac{\partial u}{\partial t} + u\cdot\nabla u = -\nabla p + \nu\Delta u.$$

Additionally, if we denote the horizontal Lagrange-Poincaré operator

$$\mathcal{LP}_{\mathrm{hor}} := \frac{D}{Dt}\left(\frac{\partial}{\partial \dot b}\right) - \frac{\partial}{\partial b}$$

we find that (just as in the inviscid case), $\mathcal{LP}_{\mathrm{hor}}(\ell_{\approx\approx}) = i_{\dot b}\widetilde{B} + F_p$, where

$$\langle F_p(b,\dot b), w_b\rangle = \int_{\partial\approx\approx_b} p(x)\langle n(x), w_b(b^{-1}(x))\rangle d^2S$$

is the pressure force on the body and

$$\ell_{\approx\approx}(b,\dot b,\xi_b) := \frac{1}{2}\int_{\approx\approx_b} \|\mathcal{I}(\dot b)(x) + \xi_b(x)\|^2 d^3x$$

is the reduced Lagrangian for the fluid. Note that $\ell = L_{\mathcal{B}} + \ell_{\approx\approx}$ and $\mathcal{LP}_{\mathrm{hor}}$ is a linear operator on the set of reduced Lagrangians. Therefore the (forced) horizontal Lagrange-Poincare equation states

$$\mathcal{LP}_{\mathrm{hor}}(L_{\mathcal{B}}) + \mathcal{LP}_{\mathrm{hor}}(\ell_{\approx\approx}) = i_{\dot{b}}\widetilde{B}_{\mu} + F_{\mathrm{ns}}.$$

Putting all these pieces together yields the forced Euler-Lagrange equation on $\mathrm{Emb}(\mathcal{B})$ given by

$$\frac{D}{Dt}\left(\frac{\partial L_{\mathcal{B}}}{\partial \dot{b}}\right) - \frac{\partial L_{\mathcal{B}}}{\partial b} = F_p + F_{\mathrm{ns}}.$$

Reversing these steps proves the equivalence of (2) and (3). □

8. CONCLUSION

In this paper we have articulated the Lagrangian picture of fluid-structure systems as systems evolving on the source fibers of a groupoid, G. This was done in a way which was remarkably parallel to the observations of V. I. Arnold in regards to fluid systems as evolving on Lie groups [Arn66]. Moreover, we have shown that the equation can be right trivialized and thus there exist equations of motion on the Lie algebroid, $\mathcal{A}$. Upon choosing a section $\mathcal{I} : T\,\mathrm{Emb}(\mathcal{B}) \hookrightarrow \mathcal{A}$, we found that these equations are the standard ones obtained via the classical theory of continuum mechanics as in [Bat00]. We were then able to derive the equations of motion for a body immersed in a Navier-Stokes fluid by adding a viscous force and a no-slip condition to our groupoid. Given this geometric perspective we can extend, specialize, and apply our findings. Such endeavors entail the future work for this research.

8.1. **Rigid bodies.** We could restrict our groupoid further by considering a subgroupoid

$$G_{\mathrm{rigid}} := \{g \in G \quad | \quad \exists A \in \mathrm{SE}(3), \beta(g) = A \cdot \alpha(g)\}.$$

We expect the resulting equations of motion to be that of a rigid body immersed in an ideal fluid. This would be a "cute" generalization of Arnold's discovery that the Euler equations are right trivialized geodesic equations on a Lie group, in that we would have found that the equations for a rigid body in an ideal fluid are right trivialized geodesic equations on a Lie groupoid. The equations of motion for a rigid body in a fluid have been treated from a geometric point of view in [Rad03, KMRMH05, VKM09, VKM10], but so far the relevance of groupoids in this context has not been explored.

8.2. **Complex fluids.** There exists a unifying framework for understanding complex fluids using the notion of advected parameters [GBR09]. This framework uses the interpretation of an incompressible fluid as an ODE on the set of volume preserving diffeomorphisms and extends the ideal fluid case by appending parameters in a vector space which are advected by the diffeomorphisms of the fluid. This framework captures magneto-hydrodynamics, micro-stretch liquid crystals,

and Yang-Mills fluids. Merging these ideas with the groupoid interpretation of fluid structure interactions presented in this paper should be possible. Moreover, the notion of "advection" fits naturally in a groupoid theoretic framework where the source map stores the initial state of the system.

8.3. **Numerical algorithms.** Discrete Lagrangian mechanics is a framework for the construction of numerical integrators for Lagrangian mechanical systems. These integrators typically preserve some of the underlying geometry, and as a consequence have good long-term conservation and stability properties (see [MW01, HLW02]). The ideas of discrete mechanics have been extended to deal with mechanical systems on Lie groups (see [LLM08] and the references therein) and in [GMP$^+$11] a variational integrator for ideal fluid dynamics was proposed by approximating $\mathrm{SDiff}(M)$ with a finite dimensional Lie group. It would be interesting to pursue a melding of these integration schemes by "discretizing" the Lie groupoid G_{rigid} to obtain a variational integrator for the interaction of an ideal fluid with a rigid body by approximating it as a finite dimensional Lie groupoid. The groupoid framework of this paper would provide a natural inroad into this problem, the more so since in [Wei95] it was shown that many variational integrators can viewed as discrete mechanical systems on Lie groupoids.

Appendix A. PRINCIPAL CONNECTIONS

The goal of this appendix is to show that the choice of a section $\mathcal{I} : T\,\mathrm{Emb}(\mathcal{B}) \to \mathcal{A}$ of the anchor map $\rho : \mathcal{A} \to T\,\mathrm{Emb}(\mathcal{B})$ defines a principal connection on the principal fiber bundle $\pi : Q(b_0) \to Q(b_0)/G(b_0)$. The material described here relies on [KN63]; see also [CMR01]. This manifestation of a section of the anchor map appeared in early work on particles immersed in fluids where such an object was called an "interpolation method" [JRD12]. For the sake of simplicity, we treat the case of an inviscid fluid; the extension to viscous flows is straightforward.

A.1. **The connection one-form.** Given a section $\mathcal{I}$ of the anchor map $\rho : \mathcal{A} \to T\,\mathrm{Emb}(\mathcal{B})$, we define a one-form A on $Q(b_0)$ with values in $\mathfrak{g}(b_0)$ by putting

$$A(b,\varphi)\cdot(\dot{b},\dot{\varphi}) = T\varphi^{-1}\circ\dot{\varphi} - \varphi^*[\mathcal{I}(\dot{b})]. \tag{29}$$

In other words, A is a map from $TQ(b_0)$ to $\mathfrak{g}(b_0)$. The right-hand side of this definition indeed gives rise to an element of $\mathfrak{g}(b_0)$, i.e. a divergence-free vector field which is tangent to the boundary of $\approx\!\!\approx_{b_0}$. To see this, note that A can be written as

$$A(b,\varphi)\cdot(\dot{b},\dot{\varphi}) = \varphi^*\left(\dot{\varphi}\circ\varphi^{-1} - \mathcal{I}(\dot{b})\right), \tag{30}$$

where the expression between parentheses on the right-hand side is an element of $\mathfrak{g}(b)$. Since φ is volume-preserving and maps the boundary of $\approx\!\!\approx_{b_0}$ to that of $\approx\!\!\approx_{b}$,

we have that the pullback vector field is an element of $\mathfrak{g}(b_0)$. We may also write this correspondence as

$$A(b,\varphi)\cdot(\dot{b},\dot{\varphi}) = \varphi^*\left(\Theta_R(\varphi,\dot{\varphi}) - \mathcal{I}(\dot{b})\right), \tag{31}$$

where Θ_R is the right Maurer-Cartan form, defined by $\Theta_R(\varphi,\dot{\varphi}) = \dot{\varphi}\circ\varphi^{-1}$. While this expression is a bit more involved than the original definition, it will make the computation of the curvature of A below easier. Strictly speaking, Θ_R is not a Maurer-Cartan form in the usual sense, but rather lives on the Lie groupoid G. It turns out, however, that Θ_R has all the properties required of it — in particular, Θ_R has zero curvature — so that we will not dwell on this point any further. For more information about groupoid Maurer-Cartan forms, see [FS08].

Proposition A.1. *The one-form A defined in* (29) *is a connection one-form.*

Proof. We need to check two properties:

(1) A is (right) equivariant under the action of $G(b_0)$ on $Q(b_0)$: for $\psi \in G(b_0)$ and $(b,\varphi,\dot{b},\dot{\varphi}) \in Q(b_0)$, we have

$$\begin{aligned} A(b,\varphi\circ\psi)\cdot(\dot{b},\dot{\varphi}\circ\psi) &= T(\varphi\circ\psi)^{-1}\circ\dot{\varphi}\circ\psi - (\varphi\circ\psi)^*[\mathcal{I}(\dot{b})] \\ &= \psi^*(T\varphi^{-1}\circ\dot{\varphi} - \mathcal{I}(\dot{b})) \\ &= \mathrm{Ad}_{\psi^{-1}}\left[A(b,\varphi)\cdot(\dot{b},\dot{\varphi})\right], \end{aligned}$$

where $\mathrm{Ad}_{\psi^{-1}} : \mathfrak{g}(b_0) \to \mathfrak{g}(b_0)$ is the adjoint action given by $\mathrm{Ad}_\psi(\xi) = T\psi\circ\xi\circ\psi^{-1}$.

(2) A maps the infinitesimal generator $\xi_{Q(b_0)}$ of an element $\xi \in \mathfrak{g}(b_0)$ back to ξ. Recall that the infinitesimal generator is the vector field $\xi_{Q(b_0)}$ on $Q(b_0)$ given by

$$\xi_{Q(b_0)}(b,\varphi) = (0, T\varphi\circ\xi) \in T_{(b,\varphi)}Q(b_0),$$

for all $\xi \in \mathfrak{g}(b_0)$ to ξ, so that

$$A(b,\varphi)\cdot\xi_{Q(b_0)} = T\varphi^{-1}\circ T\varphi\circ\xi - \mathcal{I}(0) = \xi. \qquad \square$$

Conversely, given a connection one-form A, we may define a section of the anchor map $\mathcal{I}$ as follows. For $(b,\dot{b}) \in T\,\mathrm{Emb}(\mathcal{B})$, choose φ and $\dot{\varphi}$ to be compatible with $(b,\dot{b})$, that is, so that $(b,\varphi;\dot{b},\dot{\varphi}) \in TQ(b_0)$. We then put

$$\mathcal{I}(\dot{b}) = \dot{\varphi}\circ\varphi^{-1} - \varphi_*[A(b,\varphi)\cdot(\dot{b},\dot{\varphi})]. \tag{32}$$

We first check that this prescription is well-defined, i.e. that $\mathcal{I}(b)$ does not depend on the choice of φ and $\dot{\varphi}$. Let φ' and $\dot{\varphi}'$ be different elements so that $(b,\varphi';\dot{b},\dot{\varphi}') \in TQ(b_0)$. There then exists a diffeomorphism $\psi \in G(b_0)$ and a vector field $\xi \in \mathfrak{g}(b_0)$ so that $\varphi' = \varphi\circ\psi$ and $\dot{\varphi}' = \dot{\varphi}\circ\psi + T(\varphi\circ\psi)\circ\xi$. After some calculations, we then have that

$$\dot{\varphi}'\circ\varphi'^{-1} = \dot{\varphi}\circ\varphi^{-1} + (\varphi\circ\psi)_*\xi$$

and

$$\varphi'_*[A(b,\varphi')\cdot(\dot{b},\dot{\varphi}')] = \varphi_*[A(b,\varphi)\cdot(\dot{b},\dot{\varphi})] + (\varphi\circ\psi)_*\xi,$$

so that

$$\dot{\varphi}\circ\varphi^{-1} - \varphi_*[A(b,\varphi;\dot{b},\dot{\varphi})] = \dot{\varphi}'\circ\varphi'^{-1} - \varphi'_*[A(b,\varphi';\dot{b},\dot{\varphi}')],$$

i.e. $\mathcal{I}$ is well defined.

It is not hard to show that $\mathcal{I}$ satisfies the slip boundary condition (9), so that we arrive at the following result.

Proposition A.2. *The map $\mathcal{I}$ defined in* (32) *is a section of the anchor map $\rho : \mathcal{A} \to T\,\mathrm{Emb}(\mathcal{B})$.*

A.2. **The induced horizontal bundle.** The horizontal sub-bundle induced by a connection one-form, or equivalently, by a section of ρ is defined as follows. At every point $(b,\varphi)\in Q$, we define a horizontal subspace $H(b,\varphi)\subset T_{(b,\varphi)}Q(b_0)$ by

$$H(b,\varphi) = \ker A(b,\varphi),$$

and we let $H(Q(b_0))$ be the disjoint union of all of these spaces, so that $H(Q(b_0))$ forms a sub-bundle of $TQ(b_0)$.

In our case, we have that the elements of the kernel of $A(b,\varphi)$ are the tangent vectors $(\dot{b},\dot{\varphi})$ which satisfy by (32) that $\mathcal{I}(\dot{b}) = \dot{\varphi}\circ\varphi^{-1}$. As a consequence, there exist horizontal and vertical projectors $\mathrm{hor},\mathrm{ver} : TQ(b_0)\to TQ(b_0)$ given by

$$\begin{aligned}
\mathrm{hor}(b,\varphi;\dot{b},\dot{\varphi}) &= (b,\varphi;\dot{b},\mathcal{I}(\dot{b})\circ\varphi)\\
\mathrm{ver}(b,\varphi;\dot{b},\dot{\varphi}) &= (b,\varphi;0,\dot{\varphi}-\mathcal{I}(\dot{b})\circ\varphi).
\end{aligned}$$

The horizontal bundle is therefore spanned by vectors of the form $(b,\varphi;\dot{b},\mathcal{I}(\dot{b})\circ\varphi)$, where $(b,\varphi)\in Q$ and $\dot{b}\in T_b\mathcal{B}$ are arbitrary.

A.3. **The curvature of $\mathcal{I}$.** Using Cartan's structure formula, we have that the curvature of a (right) principal connection is given by the two-form B on $Q(b_0)$ with values in $\mathfrak{g}(b_0)$, given by

$$B = \mathbf{d}A - [A,A].$$

We recall from (31) that the connection is given by $A = \Theta_R - \mathcal{I}$, and we use the fact that Θ_R has zero curvature: $\mathbf{d}\Theta_R - [\Theta_R,\Theta_R] = 0$. From this, it easily follows that the curvature B can be expressed directly in terms of $\mathcal{I}$ by

$$B = -\mathbf{d}\mathcal{I} + [\mathcal{I},\mathcal{I}].$$

Up to a sign, this is precisely the curvature tensor that appeared in Proposition (5.4).

A.4. **The induced covariant derivative.** Lastly, we use the fact that a connection one-form A induces a covariant derivative on the adjoint bundle $\tilde{\mathfrak{g}}$ to justify the expression (19) given previously. It is well-known (see e.g. [CMR01, Lemma 2.3.4]) that the covariant derivative of a curve $t \mapsto (b(t), \xi_b(t)) \in \tilde{\mathfrak{g}}$ is given by

$$\frac{D}{Dt}(b, \xi_b) = \left(b, \frac{d\xi_b}{dt} - [A(b,\varphi) \cdot (\dot{b}, \dot{\varphi}), \xi_b] \right),$$

where $(b, \varphi) \in Q(b_0)$ projects down onto $b \in \mathrm{Emb}(\mathcal{B})$, and $(\dot{b}, \dot{\varphi}) \in T_{(b,\varphi)}Q(b_0)$ is such that $\dot{\varphi} \circ \varphi^{-1} = \xi_b$. Substituting the expression (30) for A in terms of the section $\mathcal{I}$, we obtain the formula for the covariant derivative given in Proposition 5.1.

Appendix B. PROOF OF THEOREM 6.2

In this appendix, we provide a proof of Theorem 6.2. We assume that the Lagrange-Poincaré equations (23) and (24) hold. Our goal is to show that these equations yield the same dynamics as equations (26) and (25). This proof is rather long and it is easy to get lost, so we will begin with a roadmap.

Roadmap.

STEP 1. We begin by proving that the vertical equations (24) are equivalent to the inviscid fluid equations (26).

STEP 2. To prove the equivalence between the horizontal equations (23) and the body equations (25), we note that the horizontal operator

$$\mathcal{LP}_{\mathrm{hor}}(\cdot) := \frac{d}{dt}\left(\frac{\partial(\cdot)}{\partial \dot{b}}\right) - \frac{\partial(\cdot)}{\partial b}$$

is linear on vector space of real valued functions of $T\,\mathrm{Emb}(\mathcal{B}) \oplus V(\mathcal{A})$. Therefore the right hand side of the horizontal LP-equation may be written as

$$\mathcal{LP}_{\mathrm{hor}}(L_{\mathcal{B}}) + \mathcal{LP}_{\mathrm{hor}}(\ell_{\approx\approx}) = \tilde{B}_\mu(\dot{b}, \cdot). \tag{33}$$

STEP 3. The main technical part of the proof consists of computing $\mathcal{LP}_{\mathrm{hor}}(\ell_{\approx\approx})$. To do this, we proceed as follows:

step 3a. We first compute $\frac{\partial \ell_{\approx\approx}}{\partial b}$;

step 3b. We then compute $\frac{D}{Dt}\left(\frac{\partial \ell_{\approx\approx}}{\partial \dot{b}}\right)$;

step 3c. We finally observe that the Lagrange-Poincare equations can be written as $\mathcal{LP}_{\mathrm{hor}}(\ell_{\approx\approx}) = F + \tilde{B}_\mu(\dot{b}, \cdot)$ for a judiciously chosen F.

STEP 4. By the linearity of $\mathcal{LP}_{hor}$, subtracting the previous computation from the Lagrange-Poincaré equations (33) implies that

$$\mathcal{LP}_{\mathrm{hor}}(L_{\mathcal{B}}) = F.$$

STEP 5. Lastly, we prove that $F \equiv F_p$, where F_p is the boundary force defined in (27). This concludes the proof of the equivalence between (23) and (25).

Step 1. We begin with step (1) in this roadmap. It is simple to verify that

$$\left\langle \frac{\partial \ell}{\partial \xi_b}, \delta\xi_b \right\rangle = \int_{\approx_b} \langle \xi_b + I(\dot{b}), \delta\xi_b\rangle d\,\mathrm{vol} = \int_{\approx_b} \langle u, \delta\xi_b\rangle d\,\mathrm{vol}.$$

In other words, $\frac{\partial \ell}{\partial \xi_b} = u^\flat$. By equation 20 we see that

$$\frac{Du^\flat}{Dt} = \frac{\partial u^\flat}{\partial t} + \mathrm{ad}^*_{\mathcal{I}(\dot{b})}(u^\flat).$$

If we substitute these identities into the vertical LP equation we get

$$\frac{\partial u^\flat}{\partial t} + \mathrm{ad}^*_{\mathcal{I}(\dot{b})}(u^\flat) = -\,\mathrm{ad}^*_{(b,\xi_b)}(u^\flat).$$

which implies

$$\frac{\partial u^\flat}{\partial t} + \mathrm{ad}^*_u(u^\flat) = 0.$$

The last line is Arnold's description of the Euler equation $\frac{\partial u}{\partial t} + u\cdot\nabla u = \nabla p$ [AK92, §1.5] and completes step (1) of the roadmap.

Step 2. We can now move to step (2) which deals with the Lagrange-Poincaré operator. By inspection we can see that $\mathcal{LP}_{\mathrm{hor}}$ is linear and so the horizontal LP equation can be written as

$$\mathcal{LP}(L_{\mathcal{B}}) + \mathcal{LP}(\ell_{\approx}) = \tilde{B}_\mu(\dot{b}, \cdot).$$

where $\ell_{\approx}$ is the kinetic energy of the fluid, so that $\ell = L_{\mathcal{B}} + \ell_{\approx}$. Moreover, note that $\frac{\partial \ell}{\partial b} = \frac{\partial \ell_{\approx}}{\partial b} + \frac{\partial L_{\mathcal{B}}}{\partial b}$.

Step 3. This brings us to step (3a) of the roadmap, which is the computation of $\frac{\partial \ell_{\approx}}{\partial b}$. By definition $\frac{\partial \ell_{\approx}}{\partial b}(b, \dot{b}, \xi_b) \in T^*_b \,\mathrm{Emb}(\mathcal{B})$ is given by

$$\left\langle \frac{\partial \ell_{\approx}}{\partial b}(b, \dot{b}, \xi), \delta b \right\rangle = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \ell_{\approx}(b_\epsilon, \dot{b}_\epsilon, \xi_{b,\epsilon})$$

where $(b_\epsilon, \dot{b}_\epsilon, \xi_{b,\epsilon})$ is an ϵ dependent curve in $T\,\mathrm{Emb}(\mathcal{B}) \oplus V(\mathcal{A})$ such that $\delta b = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} b_\epsilon$ and

$$\frac{D}{D\epsilon}(b_\epsilon, \dot{b}_\epsilon, \xi_{b,\epsilon}) = 0.$$

By equation (19) this implies

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \xi_{b,\epsilon} = -[\mathcal{I}(\delta b), \xi_b]. \tag{34}$$

By the Reynolds transport theorem we find that

$$
\begin{aligned}
\left\langle \frac{\partial l_{\approx}}{\partial b}(b,\dot{b},\xi), \delta b \right\rangle &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \left(\frac{1}{2} \int_{\approx_{b_\epsilon}} \|\xi_{b,\epsilon} + \mathcal{I}(\dot{b}_\epsilon)\|^2 d\,\mathrm{vol} \right) \\
&= \frac{1}{2} \int_{\partial \approx_b} \|\xi_b + \mathcal{I}(\dot{b})\|^2 \mathcal{I}(\delta b) \cdot \hat{n}_b dA \\
&\quad + \frac{1}{2} \int_{\approx_b} \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \left(\|\xi_{b,\epsilon} + \mathcal{I}(\dot{b}_\epsilon)\|^2 \right) d\,\mathrm{vol}.
\end{aligned} \tag{35}
$$

Here we have used the fact that the boundary of $\approx_{b_\epsilon}$ moves with velocity $\mathcal{I}(\delta b)$ as we vary ϵ from 0. We can then focus on the second term in the above sum and apply the covariant derivative on $T^* \operatorname{Emb}(\mathcal{B}) \oplus V(\mathcal{A})^*$ to find

$$
\begin{aligned}
\frac{1}{2} \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \left(\|\xi_{b,\epsilon}(x) + \mathcal{I}(\dot{b}_\epsilon)\|^2 \right) &= \left\langle \xi_b(x) + \mathcal{I}(\dot{b})(x), \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} (\xi_{b,\epsilon}(x) + \mathcal{I}(\dot{b}_\epsilon)(x)) \right\rangle \\
&= \left\langle \xi_b(x) + \mathcal{I}(\dot{b})(x), -[\xi_b, \mathcal{I}(\delta b)] + \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} (\mathcal{I}(\dot{b}_\epsilon)) \right\rangle \\
&= \left\langle \xi_b(x) + \mathcal{I}(\dot{b})(x), -[\xi_b, \mathcal{I}(\delta b)] + \left\langle \frac{\partial \mathcal{I}}{\partial b}(\dot{b}), \delta b \right\rangle + \mathcal{I}\left(\frac{D\dot{b}}{D\epsilon} \right) \right\rangle
\end{aligned}
$$

where the second line is obtained from the first line via equation (34) and the third line is gained by viewing $\mathcal{I}$ as a vector-valued one-form and invoking proposition 2.3.

Because $\dot{b}_\epsilon$ is a parallel translate of $\dot{b}$ we know that $\frac{D\dot{b}_\epsilon}{D\epsilon} = 0$. This allows us to drop one term in the last line. Substitution into equation (35) yields

$$
\begin{aligned}
\left\langle \frac{\partial l_{\approx}}{\partial b}(b,\dot{b},\xi), \delta b \right\rangle &= \frac{1}{2} \int_{\partial \approx_b} \|\xi_b(x) + \mathcal{I}(\dot{b})(x)\|^2 \mathcal{I}(\delta b) \cdot \hat{n}_b dA \\
&\quad + \int_{\approx_b} \left\langle \xi_b(x) + \mathcal{I}(\dot{b})(x), -[\xi_b, \mathcal{I}(\delta b)] + \left\langle \frac{\partial \mathcal{I}}{\partial b}(\dot{b}), \delta b \right\rangle \right\rangle d\,\mathrm{vol}
\end{aligned}
$$

Finally, we may reduce clutter by substituting $u = \xi_b + \mathcal{I}(\dot{b})$ and invoking the divergence theorem to transform the surface integral into a volume integral. This yields

$$
\begin{aligned}
\left\langle \frac{\partial \ell_{\approx}}{\partial b}(b,\dot{b},\xi_b), \delta b \right\rangle &= \int_{\approx_b} \left\{ \operatorname{div}\left(\frac{1}{2} \|u\|^2 \mathcal{I}(\delta b) \right) \right. \\
&\quad \left. + \left\langle u, [\mathcal{I}(\delta b), \xi_b] + \left\langle \left.\frac{\partial \mathcal{I}}{\partial b}\right|_{\dot{b}}, \delta b \right\rangle \right\rangle \right\} d\,\mathrm{vol}
\end{aligned} \tag{36}
$$

This completes step (3a) and we may move to step (3b). We find that the fiber derivative of $\ell_{\approx\approx}$ is given by

$$
\begin{aligned}
\left\langle \frac{\partial \ell_{\approx\approx}}{\partial \dot{b}}(b, \dot{b}, \xi_b), \delta b \right\rangle &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \ell_{\approx\approx}(b, \dot{b} + \epsilon \delta b, \xi_b) \\
&= \int_{\approx\approx_b} \left\langle \xi_b + \mathcal{I}(\dot{b}), \mathcal{I}(\delta b) \right\rangle d\,\mathrm{vol} \\
&= \langle u^\flat, \mathcal{I}(\delta b) \rangle.
\end{aligned}
$$

By the definition of the covariant derivative on $V(\mathcal{A})^*$ we find

$$
\left\langle \frac{D}{Dt}\left(\frac{\partial \ell_{\approx\approx}}{\partial \dot{b}}\right), \delta b \right\rangle = \underbrace{\frac{d}{dt}\left(\left\langle \frac{\partial \ell_{\approx\approx}}{\partial \dot{b}}, \delta b \right\rangle\right)}_{T_1} - \underbrace{\left\langle \frac{\partial \ell_{\approx\approx}}{\partial \dot{b}}, \frac{D\delta b}{Dt} \right\rangle}_{T_2} \tag{37}
$$

We can see by direct computation that

$$
T_2 = \int_{\approx\approx_b} \left\langle u, \mathcal{I}\left(\frac{D\delta b}{Dt}\right) \right\rangle d\,\mathrm{vol}
$$

This is as far as we need to go with T_2 and we will now rework T_1. Firstly, we note that T_1 is the time derivative of an integral over a time dependent domain, so we must invoke the Reynolds transport theorem a second time. This yields the equivalence

$$
T_1 = \int_{\partial \approx\approx_b} \langle \xi_b + \mathcal{I}(\dot{b}), \mathcal{I}(\delta b) \rangle \langle \xi_b + \mathcal{I}(\dot{b}), \hat{n}_b \rangle dA + \int_{\approx\approx_b} \frac{d}{dt} \langle \xi_b + \mathcal{I}(\dot{b}), \mathcal{I}(\delta b) \rangle d\mathrm{vol}
$$

By construction $\langle \xi_b, \hat{n}_b \rangle = 0$ on the boundary of $\approx\approx_b$ so that

$$
T_1 = \int_{\partial \approx\approx_b} \langle \xi_b + \mathcal{I}(\dot{b}), \mathcal{I}(\delta b) \rangle \langle \mathcal{I}(\dot{b}), \hat{n}_b \rangle dA + \int_{\approx\approx_b} \frac{d}{dt} \langle u, \mathcal{I}(\delta b) \rangle) d\,\mathrm{vol}.
$$

Finally substituting $u = \mathcal{I}(\dot{b}) + \xi_b$ yields

$$
T_1 = \int_{\partial \approx\approx_b} \langle u, \mathcal{I}(\delta b) \rangle \langle \mathcal{I}(\dot{b}), \hat{n}_b \rangle dA + \int_{\approx\approx_b} \langle \frac{\partial u}{\partial t}, \mathcal{I}(\delta b) \rangle + \langle u, \frac{\partial}{\partial t} \mathcal{I}(\delta b) \rangle d\,\mathrm{vol}.
$$

The term $\frac{\partial}{\partial t}\mathcal{I}(\delta b)$ can be handled by invoking proposition 2.3 a second time. Explicitly, this give us the equivalence

$$
\frac{\partial}{\partial t} \mathcal{I}(\delta b) = \left\langle \frac{\partial \mathcal{I}}{\partial b}(\delta b), \dot{b} \right\rangle + \mathcal{I}\left(\frac{D\delta b}{Dt}\right)
$$

where we have used the fact that $\mathcal{I}$ is fiberwise linear on $T\,\mathrm{Emb}(\mathcal{B})$, and therefore $\mathcal{I}$ is equal to its own fiber derivative. We can substitute the above computation

into T_1 to get the final expression

$$T_1 = \int_{\partial \approx\approx_b} \langle u, \mathcal{I}(\delta b)\rangle \langle \mathcal{I}(\dot{b}), \hat{n}_b\rangle dA \\ + \int_{\approx\approx_b} \left\langle \frac{\partial u}{\partial t}, \mathcal{I}(\delta b)\right\rangle + \left\langle u, \left\langle \frac{\partial \mathcal{I}}{\partial b}(\delta b), \dot{b}\right\rangle + \mathcal{I}\left(\frac{D\delta b}{Dt}\right)\right\rangle d\,\mathrm{vol}$$

We can finally revisit equation (37), and write it as

$$\begin{aligned}\left\langle \frac{D}{Dt}\left(\frac{\partial \ell_{\approx\approx}}{\partial \dot{b}}\right), \delta b\right\rangle &= T_1 - T_2 \\ &= \int_{\partial \approx\approx_b} \langle u, \mathcal{I}(\delta b)\rangle \langle \mathcal{I}(\dot{b}), \hat{n}_b\rangle dA \\ &\quad + \int_{\approx\approx_b} \left\langle \frac{\partial u}{\partial t}, \mathcal{I}(\delta b)\right\rangle + \left\langle u, \left\langle \frac{\partial \mathcal{I}}{\partial b}(\delta b), \dot{b}\right\rangle\right\rangle d\,\mathrm{vol}\end{aligned}$$

We can substitute the vertical equation $\frac{\partial u}{\partial t} = \nabla p - u \cdot \nabla u$ and (as is customary) we will convert the surface integral into a volume integral via the divergence theorem. This yields

$$\begin{aligned}\left\langle \frac{D}{Dt}\left(\frac{\partial \ell_{\approx\approx}}{\partial \dot{b}}\right), \delta b\right\rangle &= \int_{\approx\approx_b} \Big\{ \mathrm{div}\left(\langle u, \mathcal{I}(\delta b)\rangle \mathcal{I}(\dot{b})\right) \\ &\quad + \langle \nabla p - u \cdot \nabla u, \mathcal{I}(\delta b)\rangle + \left\langle u, \left\langle \frac{\partial \mathcal{I}}{\partial b}(\delta b), \dot{b}\right\rangle\right\rangle \Big\} d\,\mathrm{vol}\end{aligned} \tag{38}$$

This bring us to step (3c) where we calculate $\mathcal{LP}_{\mathrm{hor}}(\ell_{\approx\approx})$. By subtracting (36) from (38) we find

$$\begin{aligned}\frac{D}{Dt}\left(\frac{\partial \ell_{\approx\approx}}{\partial \dot{b}}\right) - \frac{\partial \ell_{\approx\approx}}{\partial b} &= \int_{\approx\approx_b} \Big\{ \mathrm{div}\left(\langle u, \mathcal{I}(\delta b)\rangle \mathcal{I}(\dot{b}) - \frac{1}{2}\|u\|^2 \mathcal{I}(\delta b)\right) \\ &\quad + \langle \nabla p - u \cdot \nabla u, \mathcal{I}(\delta b)\rangle + \langle u, [\mathcal{I}(\delta b), \xi_b]\rangle \\ &\quad + \langle u, \left\langle \frac{\partial \mathcal{I}}{\partial b}(\delta b), \dot{b}\right\rangle - \left\langle \frac{\partial \mathcal{I}}{\partial b}(\dot{b}), \delta b\right\rangle \Big\} d\,\mathrm{vol}\end{aligned}$$

We may now invoke proposition 2.4 to replace the term $\left\langle \frac{\partial \mathcal{I}}{\partial b}(\delta b), \dot{b}\right\rangle - \left\langle \frac{\partial \mathcal{I}}{\partial b}(\dot{b}), \delta b\right\rangle$ with $d\mathcal{I}(\dot{b}, \delta b)$. Finally we use the equivalence $B(\dot{b}, \delta b) = d\mathcal{I}(\dot{b}, \delta b) + [\mathcal{I}(\dot{b}), \mathcal{I}(\delta b)]$ to rewrite the last equation as

$$\begin{aligned}\frac{D}{Dt}\left(\frac{\partial \ell_{\approx\approx}}{\partial \dot{b}}\right) - \frac{\partial \ell_{\approx\approx}}{\partial b} &= \int_{\approx\approx_b} \Big\{ \mathrm{div}\left(\langle u, \mathcal{I}(\delta b)\rangle \mathcal{I}(\dot{b}) - \frac{1}{2}\|u\|^2 \mathcal{I}(\delta b)\right) \\ &\quad + \langle \nabla p - u \cdot \nabla u, \mathcal{I}(\delta b)\rangle + \langle u, [\mathcal{I}(\delta b), \xi_b]\rangle \\ &\quad + \langle u, [\mathcal{I}(\delta b), \mathcal{I}(\dot{b})]\rangle + \langle u, B(\dot{b}, \delta b)\rangle \Big\} d\,\mathrm{vol}\end{aligned}$$

We define the term $B_\mu(\dot{b}, \delta b) = \int_{\approx\!\approx_b} \langle u, B(\dot{b}, \delta b)\rangle d\,\mathrm{vol}$, so that we may conclude

$$\mathcal{LP}_{\mathrm{hor}}(\ell_{\approx\!\approx}) = F + B_\mu(\dot{b}, \cdot). \tag{39}$$

Step 4. This brings us to step (4). Given that $\mathcal{LP}_{\mathrm{hor}}$ is linear on the set of Lagrangians the horizontal LP equations give us

$$\begin{aligned} B_\mu(\dot{b}, \cdot) &= \mathcal{LP}_{\mathrm{hor}}(\ell) \\ &= \mathcal{LP}_{\mathrm{hor}}(\ell_{\approx\!\approx}) + \mathcal{LP}_{\mathrm{hor}}(L_{\mathcal{B}}) \\ &= B_\mu(\dot{b}, \cdot) + F + \mathcal{LP}_{\mathrm{hor}}(L_{\mathcal{B}}), \end{aligned}$$

so that

$$\mathcal{LP}(L_{\mathcal{B}}) = F.$$

Step 5. This brings us to step (5) where we will manipulate F. To reduce clutter, set $v = \mathcal{I}(\delta b)$ and $w = \mathcal{I}(\dot{b})$ so that the term given by F may be rewritten as

$$F = \int_{\approx\!\approx_b} \mathrm{div}\left((u \cdot v)w - \frac{1}{2}(u \cdot u)v\right) + u \cdot [v, u] + (\nabla p - u \cdot \nabla u) \cdot v\, d^3x$$

Using Einstein's summation notation we may write F as:

$$F = \int_{\approx\!\approx_b} \frac{\partial}{\partial x^k}\left(u^i v^i w^k - \frac{1}{2}u^i u^i v^k\right) + u^k\left(\frac{\partial u^k}{\partial x^i}v^i - \frac{\partial v^k}{\partial x^i}u^i\right) + \left(\frac{\partial p}{\partial x^k}v^k - u^i v^k \frac{\partial u^k}{\partial x^i}\right) d^3x.$$

By applying the divergence free properties, $\frac{\partial b^k}{\partial x^k} = \frac{\partial a^k}{\partial x^k} = 0$, this simplifies to

$$\begin{aligned} F &= \int_{\approx\!\approx_b} \frac{\partial}{\partial x^k}(v^i u^i)(w^k - u^k) + \frac{\partial p}{\partial x^k} v^k d^3x \\ &= \int_{\approx\!\approx_b} \mathrm{div}\left(p(x)w(x) - (w(x) \cdot u(x))\, \xi_b(x)\right) d^3x \\ &= \int_{\partial \approx\!\approx_b} \langle p(x)\mathcal{I}(\delta b)(x) - \langle \mathcal{I}(\delta b), u(x)\rangle \xi_b(x), \hat{n}(x)\rangle d^2x. \end{aligned}$$

Noting that $\xi_b = u - \mathcal{I}(\dot{b}) \equiv u - v$ is tangent to the boundary, and therefore orthogonal to the unit normal $\hat{n}$ we can drop the term involving ξ_b so that

$$F = \int_{\partial \approx\!\approx_b} \langle p(x)\mathcal{I}(\delta b)(x), \hat{n}(x)\rangle d^2x$$

and by the no-slip condition of $\mathcal{I}$ we find

$$F = \int_{\partial \approx\!\approx_b} \langle p(x)\hat{n}(x), (\delta b \circ b^{-1})(x)\rangle d^2x,$$

so that $F = F_p$, as claimed.

Thus we may conclude that the Euler-Lagrange equations for $L_{\mathcal{B}}$ are given by

$$\left\langle \frac{D}{Dt}\left(\frac{\partial L_{\mathcal{B}}}{\partial \dot{b}}\right) - \frac{\partial L_{\mathcal{B}}}{\partial b}, \delta b \right\rangle = \int_{\partial \approx\!\approx_b} \langle p(x)\hat{n}(x), (\delta b \circ b^{-1})(x)\rangle d^2x$$

for arbitrary variations δb. Since the variation δb is arbitrary the theorem follows.

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